1id1 Preprint | 20 November 2026
1id1:core:20261120-0002@v1 Version 1
https://1id1.net/id/core/20261120-0002/v1/

FINITE ORTHANT-ORDER SYSTEMS: COMPATIBILITY, INCOMPARABILITY, AND COUNTING

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Abstract. We study the combinatorial structure of finite orthant-order systems. For a finite set D of records in a product of $d \geq 2$ chains, independent coordinate reversals index a family of 2^d product preorders. The pairwise sign map $D \times D \to \{-, 0, +\}^d$ represents the whole system; the orientations comparing an unordered pair form two antipodal Boolean faces. For a fixed anchor q, call $A \subseteq D \setminus \{q\}$ compatible if one orientation makes every point of A comparable with q. We prove that its compatible orientations are either empty or an affine binary space. Equivalently, compatibility is balance of a signed constraint graph, and a balanced graph with c components yields exactly 2^c orientations. This gives linear-time recognition and counting, an exact Helly number d, and a root-free rank-one sign-completion model. By Dowker duality the compatibility complex has a homotopy type bounded solely by d, whereas finding a largest compatible subset is NP-hard.

Requiring every point to be incomparable with the anchor instead produces signed not-all-equal constraints. Supports of size at most two remain affine and linear-time solvable, while support three already gives NP-complete decision and #P-complete counting problems; mixed comparability and incomparability constraints have the same boundary. Antichain orientations are complements of unions of antipodal faces, yielding deterministic existence bounds. In an independent continuous product model, the number of comparable pairs under a fixed orientation has a Poisson limit at $d - 2\log_2 N \to t$, with zero-pair probability tending to $\exp(-2^{-t})$. Replacing chains by posets finally separates intrinsic from orientation-induced incomparability.

1. Introduction

The coordinatewise order on $\mathbb{R}^d$ is among the most elementary partial orders. It underlies the maxima problem in computational geometry [16], Pareto dominance, and skyline queries in databases [8]. Reversing any chosen subset of the coordinate orders gives 2^d product orders. Geometrically these choices correspond to the 2^d closed orthants about an anchor point, or to the chambers of the coordinate hyperplane arrangement [1, 6].

2020 *Mathematics Subject Classification.* Primary 06A06, 05C22; Secondary 05C15, 52C35, 68Q17, 68Q25.

Key words and phrases. product order, signed graph, incomparability, antichain, Dowker complex, Poisson approximation.

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Usually one orientation is fixed by the application: all objectives are minimised, all are maximised, or a prescribed mixture is used. In other settings all orientations occur simultaneously. Opposite orthants are studied together in discrete geometry [2]; global skyline queries combine quadrant skylines relative to a query point [17]; and qualitative representations retain the three coordinate relations $<, =, >$. The common structure is not one isolated order but a tightly coupled family indexed by the Boolean cube.

Incomparability has two distinct sources. Even for numeric coordinates, comparability in a fixed product order requires all coordinate comparisons to align; for two independent continuous random points its probability is 2^{1-d}, so high-dimensional samples rapidly become antichains. In heterogeneous data, a coordinate may itself be only partially ordered: distinct nominal labels, for example, are intrinsically incomparable unless an external hierarchy is supplied. For mixed-type learning pipelines, modelling such categories by a flat poset avoids imposing an artificial total order while retaining genuine order information from numerical or hierarchical coordinates. The orthant family separates these coordinate-level obstructions from the failure of one orientation to align a collection of otherwise comparable coordinate relations.

An early motivation for the present work appeared in [18]. There, the 2^d coordinate-reversed orders were used to construct local order graphs for instance-based learning, and the number of orders in which two records are comparable was expressed in terms of their equal coordinates. The local graph construction was phrased using cover relations. The purpose of the present paper is different: we discard cover relations and study the entire indexed order system as a finite combinatorial object.

The learning paper [19] defines the finite orthant-order system and asks how to predict comparability when the orientation is hidden. Here we fix a record q and vary the subset of records required to be comparable with it. The link is exact: the orientations compatible with A are the positive version space of the feature-vector pairs $\{\varphi(q), \varphi(x)\}$, $x \in A$, after dropping identical-endpoint pairs, which impose no constraint. We use the shared sign representation, balance criterion, Helly bound and counting results as foundations. Our focus is the structure of the compatible subsets themselves: their minimal obstructions, their facets and homotopy type, and what changes when a coordinate is only partially ordered.

There is an immediate reason that the indexing cannot be forgotten. For every pair $x, y \in \mathbb{R}^d$, some orientation makes x below y. Consequently, the union of all 2^d order relations is the universal relation $D \times D$. The structure is carried by *which* orientations compare a pair, not by the unlabelled union.

The results are organised into four groups.

(i) We give an exact sign representation of the 2^d coordinate orientations. For an anchor q, the compatible orientations form either the empty set or a coset of an explicit binary subspace. A signed graph and a root-free bipartite incidence graph both recognise compatibility by

balance; if the coordinate graph has c components, the accord has exactly 2^c orientations. Recognition, counting, and certification are linear in the sign representation.

(ii) Negative cycles yield point-minimal obstructions of arbitrary size and the exact Helly number d, dropping to two in general position. Dowker duality identifies the compatibility complex, up to homotopy, with a complex on at most 2^{d-1} vertices. Thus its homotopy type is bounded by d, although finding a largest compatible subset is NP-hard.

(iii) Anchored incomparability is a signed not-all-equal constraint system. Binary supports give an affine linear-time problem, while ternary supports give NP-complete decision and #P-complete counting problems. An inclusion–exclusion identity recovers the count from accord numbers, and mixed comparability–incomparability systems obey the same boundary.

(iv) Antichain orientations are complements of unions of antipodal Boolean faces. We derive deterministic existence and counting bounds and, in a random-product model, a Poisson limit with an explicit threshold profile. Extending the sign alphabet to products of posets then separates intrinsic coordinate incomparability from orientation-induced incomparability and supplies a canonical model for categorical coordinates.

The results depend only on coordinate comparisons. They are therefore invariant under arbitrary strictly monotone transformations of individual coordinates, including independent reversals. This covariance is the reason for treating the complete orientation family as one object.

2. Orthant-order systems

2.1. **Products of chains and coordinate reversals.** Throughout, $d \geq 2$. Let $[d] = \{1, \dots, d\}$, and let

$$X = X_1 \times \cdots \times X_d$$

be a product of chains. A finite data set is a finite set D equipped with a map $\varphi\colon D \to X$. We write $\varphi(x) = (x_1, \dots, x_d)$ and allow distinct labelled elements of D to have the same coordinate vector.

Let

$$\Sigma_d = \{-, +\}^d$$

be the set of orientations. We multiply signs in the usual way, identifying $+$ with 1 and $-$ with -1.

Definition 2.1 (Oriented product preorder)**.** For $\sigma = (\sigma_1, \dots, \sigma_d) \in \Sigma_d$, define $\preceq_\sigma$ on D by

$$x \preceq_\sigma y \quad \Longleftrightarrow \quad \begin{cases} x_i \leq y_i, & \sigma_i = +, \\ x_i \geq y_i, & \sigma_i = -, \end{cases} \qquad \text{for every } i \in [d].$$

The indexed family

$$\mathcal{O}(D) = (D, (\preceq_\sigma)_{\sigma \in \Sigma_d})$$

is the *finite orthant-order system* induced by D.

Each $\preceq_\sigma$ is a preorder. It is a partial order whenever φ is injective. In general, let

$$x \equiv y \quad \Longleftrightarrow \quad x_i = y_i \text{ for every } i.$$

Then every $\preceq_\sigma$ induces a partial order on $D/\equiv$. Thus duplicated records create equivalence classes, not a failure of the order construction.

The group $\Sigma_d \cong C_2^d$ acts on itself by coordinatewise multiplication. Write $-\sigma$ for the antipodal orientation.

Proposition 2.2 (Duality)**.** *For every $\sigma \in \Sigma_d$,*

$$x \preceq_\sigma y \quad \Longleftrightarrow \quad y \preceq_{-\sigma} x.$$

Consequently, $\preceq_{-\sigma}$ is the opposite preorder of $\preceq_\sigma$.

Proof. Reversing every coordinate order reverses every coordinate comparison. □

If direction is retained, there are 2^d indexed preorders. If an orientation is identified with its opposite and only unordered comparability is considered, there are 2^{d-1} orientation pairs.

2.2. **The sign representation.** Let $\{-, 0, +\}^d$ be ordered by the face order in which

$$0 < +, \qquad 0 < -,$$

coordinatewise, while $+$ and $-$ are incomparable. For $v \in \{-, 0, +\}^d$, write

$$\operatorname{supp}(v) = \{i \in [d] : v_i \neq 0\}$$

and define the Boolean face

$$F(v) = \{\sigma \in \Sigma_d : \sigma_i = v_i \text{ for all } i \in \operatorname{supp}(v)\}.$$

Thus $F(v)$ has $2^{d-|\operatorname{supp}(v)|}$ vertices.

Definition 2.3 (Pairwise sign map)**.** For $x, y \in D$, define

$$\chi(x, y)_i = \begin{cases} +, & x_i < y_i, \\ 0, & x_i = y_i, \\ -, & x_i > y_i. \end{cases}$$

The map $\chi\colon D \times D \to \{-, 0, +\}^d$ is the *pairwise sign representation* of $\mathcal{O}(D)$.

We have $\chi(y, x) = -\chi(x, y)$ and $\chi(x, x) = 0$.

Theorem 2.4 (Sign representation)**.** *For all $x, y \in D$,*

$$\{\sigma \in \Sigma_d : x \preceq_\sigma y\} = F(\chi(x, y)).$$

Hence the indexed family $(\preceq_\sigma)_{\sigma \in \Sigma_d}$ and the sign map χ determine one another.

Proof. If $x_i < y_i$, then $x \preceq_\sigma y$ requires the original order in coordinate i, so $\sigma_i = +$. If $x_i > y_i$, it requires the dual order, so $\sigma_i = -$. An equality imposes no condition on σ_i. These are exactly the defining conditions of $F(\chi(x,y))$.

Conversely, the face $F(\chi(x,y))$ records every nonzero coordinate of $\chi(x,y)$ as a fixed sign; a coordinate is zero precisely when both signs occur in that position among the vertices of the face. Thus the indexed comparison sets recover χ. □

Definition 2.5 (Comparability orientation set)**.** For $x, y \in D$, let

$$C(x,y) = \{\sigma \in \Sigma_d : x \preceq_\sigma y \text{ or } y \preceq_\sigma x\}.$$

Corollary 2.6 (Antipodal-face representation)**.** *For all $x, y \in D$,*

$$C(x,y) = F(\chi(x,y)) \cup F(-\chi(x,y)).$$

If $x \not\equiv y$, the two faces are disjoint and antipodal.

Proof. Apply Theorem 2.4 to (x,y) and (y,x). When $\chi(x,y) \neq 0$, a sign vector cannot agree with both $\chi(x,y)$ and $-\chi(x,y)$ on their nonempty common support. □

Proposition 2.7 (Pairwise orientation count)**.** *Let*

$$e(x,y) = |\{i \in [d] : x_i = y_i\}|.$$

Then

$$|C(x,y)| = 2^{\min\{e(x,y)+1,d\}}.$$

Proof. If $x \not\equiv y$, each face in Theorem 2.6 has $2^{e(x,y)}$ vertices and the faces are disjoint. If $x \equiv y$, both faces equal Σ_d, which has 2^d vertices. □

The count in Theorem 2.7 is the order-system version of a property used in the local graph construction of [18]. The general counting theorem below replaces a pair by an arbitrary family of points.

Remark 2.8 (The union is uninformative)*.* For every ordered pair (x,y), the face $F(\chi(x,y))$ is nonempty. Therefore

$$\bigcup_{\sigma \in \Sigma_d} \preceq_\sigma = D \times D.$$

The orientation labels, or equivalently the sign representation, are essential.

3. Compatibility about an anchor

Fix an anchor $q \in D$. For $x \in D$, abbreviate

$$C_q(x) = C(q,x).$$

Definition 3.1 (Orientation accord and compatibility)**.** For $A \subseteq D \setminus \{q\}$, the *orientation accord* of A about q is

$$\mathrm{Acc}_q(A) = \bigcap_{x \in A} C_q(x) \subseteq \Sigma_d.$$

We call A *orthant-compatible about* q if $\mathrm{Acc}_q(A) \neq \varnothing$. Its *accord number* is

$$a_q(A) = |\,\mathrm{Acc}_q(A)|.$$

For $\sigma \in \Sigma_d$, define the closed σ-orthant at q by

$$O_\sigma(q) = \{x \in D : q \preceq_\sigma x\}.$$

Then A is compatible about q exactly when

$$A \subseteq O_\sigma(q) \cup O_{-\sigma}(q)$$

for some σ. The direction of comparison with q may differ from one point to another.

For $x \neq q$, let

$$S_q(x) = \{i \in [d] : x_i \neq q_i\}$$

and let $\alpha_q(x) \in \{-,+\}^{S_q(x)}$ be the nonzero part of $\chi(q,x)$. Thus $\alpha_q(x)_i$ records on which side of q_i the coordinate x_i lies.

Lemma 3.2 (One-point compatibility)**.** *An orientation σ lies in $C_q(x)$ if and only if there exists $\tau_x \in \{-,+\}$ such that*

$$\sigma_i = \tau_x \alpha_q(x)_i \qquad \text{for every } i \in S_q(x).$$

Equivalently,

$$\sigma_i \sigma_j = \alpha_q(x)_i \alpha_q(x)_j \qquad \text{for all } i,j \in S_q(x).$$

Proof. By Theorem 2.6, $\sigma \in C_q(x)$ precisely when it agrees on $S_q(x)$ either with $\alpha_q(x)$ or with its antipode. This gives the first statement. Eliminating the common sign τ_x gives the second. Conversely, the pairwise equations imply that $\sigma_i \alpha_q(x)_i$ is constant on $S_q(x)$. □

The second formulation is the key: a point does not select a single orientation. It imposes parity constraints between those coordinates on which it differs from the anchor.

4. Binary and signed-graph representations

4.1. **Affine binary constraints.** Identify Σ_d with $\mathbb{F}_2^d$ by

$$+ \longleftrightarrow 0, \qquad - \longleftrightarrow 1.$$

Under this identification, multiplication of signs becomes addition modulo two. Write b_i for the binary variable representing σ_i, and $\beta_q(x)_i$ for the binary representation of $\alpha_q(x)_i$. By Theorem 3.2, x imposes

$$b_i + b_j = \beta_q(x)_i + \beta_q(x)_j \pmod 2, \qquad i,j \in S_q(x).$$

Proposition 4.1 (Coset structure)**.** *For $x \in D \setminus \{q\}$, let*

$$W_x = \{b \in \mathbb{F}_2^d : b_i = b_j \text{ for all } i,j \in S_q(x)\} = \langle e_i : i \notin S_q(x)\rangle + \langle \mathbf{1}\rangle.$$

Then $C_q(x)$, viewed in $\mathbb{F}_2^d$, is a coset of W_x. Consequently, for every $A \subseteq D \setminus \{q\}$, the orientation accord $\mathrm{Acc}_q(A)$ is either empty or a coset of

$$W_A = \bigcap_{x \in A} W_x,$$

and $\dim_{\mathbb{F}_2} W_A$ equals the number of connected components of the support hypergraph $([d], \{S_q(x) : x \in A\})$, isolated vertices included. Since $\mathbf{1} \in W_x$ for every x, a nonempty accord is invariant under translation by the all-ones vector $\mathbf{1}$, that is, under the antipodal map $\sigma \mapsto -\sigma$.

Proof. Let $\widetilde{\beta} \in \mathbb{F}_2^d$ agree with $\beta_q(x)$ on $S_q(x)$ and vanish elsewhere. By Theorem 3.2, b represents an orientation in $C_q(x)$ if and only if $b + \widetilde{\beta}$ is constant on $S_q(x)$, that is, $b \in \widetilde{\beta} + W_x$. A nonempty intersection of cosets is a coset of the intersection of the underlying subspaces, which gives the statement about $\mathrm{Acc}_q(A)$.

A vector lies in W_A precisely when it is constant on every support $S_q(x)$, hence precisely when it is constant on every connected component of the support hypergraph. Choosing one bit per component is a linear isomorphism, so the dimension is the number of components. The equality $W_x = \langle e_i : i \notin S_q(x) \rangle + \langle \mathbf{1} \rangle$ is immediate, and $\mathbf{1} \in W_x$ gives the antipodal invariance. □

Thus accord numbers are either zero or powers of two, and the exponent is already identified: it is the number of components of the support hypergraph. What the coset picture does not provide is a feasibility criterion and a certificate of failure. For these we pass to a signed graph, which is also the sparse data structure used by the algorithm of Theorem 9.1.

Remark 4.2 (The folded cube). Every constraint set in this paper is invariant under the antipodal map, so the theory descends to the quotient group $\mathbb{F}_2^d/\langle \mathbf{1} \rangle \cong \mathbb{F}_2^{d-1}$, whose elements are the antipodal orientation pairs. There each $C_q(x)$ becomes a single coset of $W_x/\langle \mathbf{1} \rangle$: anchored compatibility is a coset-intersection problem, and the anchored incomparability of Section 6 is the complementary coset-covering problem, for one and the same family of cosets in the folded cube. The Dowker duality of Theorem 5.10 lives naturally on this quotient.

4.2. **The signed constraint graph.** A signed multigraph is a multigraph whose edges have labels in $\{-, +\}$. The sign of a cycle is the product of its edge signs. A signed graph is *balanced* if every cycle is positive [14, 22]. Parallel edges of opposite signs form a negative cycle of length two.

Definition 4.3 (Signed constraint graph)**.** Let $A \subseteq D \setminus \{q\}$. The signed constraint graph $G_q(A)$ has vertex set $[d]$. For each $x \in A$ with $|S_q(x)| \geq 2$, choose a root $r_x \in S_q(x)$. For every $i \in S_q(x) \setminus \{r_x\}$, add an edge $r_x i$ with sign

$$\lambda_x(r_x i) = \alpha_q(x)_{r_x} \alpha_q(x)_i.$$

The graph may depend on the selected roots, but its signing solution set does not. A vertex signing $\sigma \in \Sigma_d$ *satisfies* $G_q(A)$ if

$$\sigma_i \sigma_j = \lambda(ij)$$

for every signed edge ij.

Lemma 4.4 (Constraint equivalence)**.** *An orientation σ belongs to* $\mathrm{Acc}_q(A)$ *if and only if it satisfies every edge of $G_q(A)$.*

Proof. For a fixed point x, the star edges in Theorem 4.3 state

$$\sigma_{r_x}\sigma_i = \alpha_q(x)_{r_x}\alpha_q(x)_i \qquad (i \in S_q(x) \setminus \{r_x\}).$$

They imply $\sigma_i/\alpha_q(x)_i = \sigma_{r_x}/\alpha_q(x)_{r_x}$ throughout $S_q(x)$, which is the condition in Theorem 3.2. The converse is immediate. □

4.3. **Balance, compatibility, and counting.**

Theorem 4.5 (Balance–compatibility–counting theorem)**.** *Let $A \subseteq D \setminus \{q\}$, and let $c_q(A)$ be the number of connected components of $G_q(A)$, including isolated coordinate vertices. The following are equivalent:*

(a) A is orthant-compatible about q;
(b) $G_q(A)$ admits a satisfying vertex signing;
(c) $G_q(A)$ is balanced.

If these conditions hold, then

$$\mathrm{Acc}_q(A) \cong \mathbb{F}_2^{c_q(A)} \quad \text{as an affine space,}$$

and consequently

$$a_q(A) = 2^{c_q(A)}.$$

After identifying antipodal orientations, the number of compatible orientation pairs is $2^{c_q(A)-1}$.

Proof. The equivalence of (a) and (b) is Theorem 4.4.

Suppose (b) holds, and let σ satisfy the edges. Consider a cycle with vertex sequence

$$v_0, v_1, \ldots, v_{k-1}, v_k = v_0.$$

The product of its edge signs is

$$\prod_{j=0}^{k-1} \sigma_{v_j}\sigma_{v_{j+1}} = +,$$

because every vertex sign appears twice. Hence the graph is balanced.

Conversely, suppose the graph is balanced. Choose one root vertex in each connected component and assign it an arbitrary sign. Propagate signs along a spanning tree using $\sigma_j = \lambda(ij)\sigma_i$. If a non-tree edge uv were violated, the edge together with the tree path from u to v would form a negative cycle. Balance therefore guarantees a satisfying signing.

There are two independent choices for the root sign in each of the $c_q(A)$ components, and all other signs are then forced. This yields exactly $2^{c_q(A)}$

satisfying orientations. Global sign reversal acts freely and pairs the solutions, giving $2^{c_q(A)-1}$ antipodal pairs. Alternatively, the components of $G_q(A)$ coincide with the components of the support hypergraph, so the count also follows from Theorem 4.1. □

Corollary 4.6 (The pair count revisited)**.** *For $A = \{x\}$, Theorem 4.5 reduces to*

$$a_q(\{x\}) = 2^{\min\{e(q,x)+1,d\}},$$

the formula of Theorem 2.7.

Proof. If $x \not\equiv q$, the constraint graph consists of one tree on $S_q(x)$ and $e(q,x)$ isolated vertices, hence has $e(q,x)+1$ components. If $x \equiv q$, it has d isolated vertices. □

Corollary 4.7 (General position at the anchor)**.** *Suppose $x_i \neq q_i$ for every $x \in A$ and $i \in [d]$. Then A is compatible about q if and only if the sign vectors $\chi(q,x)$, $x \in A$, are all equal up to antipodes. When A is nonempty and compatible, it has exactly one compatible antipodal orientation pair.*

Proof. Every point has full support, so its constraints connect all coordinate vertices. Compatibility forces all full sign vectors to agree up to a global sign. The constraint graph is connected, so Theorem 4.5 gives $2^{1-1} = 1$ antipodal pair. □

Remark 4.8 (Cohomological reading)*.* Identify signs with $\mathbb{F}_2$ and regard the edge signs of $G_q(A)$ as a 1-cochain $\lambda \in C^1(G_q(A);\mathbb{F}_2)$. A vertex signing $\sigma \in C^0$ satisfies the graph exactly when $\delta\sigma = \lambda$, where δ is the coboundary map. Hence A is compatible about q if and only if the class $[\lambda] \in H^1(G_q(A);\mathbb{F}_2)$ vanishes, and in that case $\mathrm{Acc}_q(A)$ is a torsor over $H^0(G_q(A);\mathbb{F}_2) \cong \mathbb{F}_2^{c_q(A)}$. Evaluating λ on the cycle space detects the class; a negative cycle is exactly a cycle on which the evaluation is nonzero. Theorem 4.5 is thus the exactness of $0 \to H^0 \to C^0 \xrightarrow{\delta} C^1 \to H^1 \to 0$ read through the orthant dictionary.

Remark 4.9 (Relation to signed graphs and learning)*.* The equivalence between balance and the existence of a satisfying vertex signing is classical signed-graph theory [14, 15]. The content specific to orthant-order systems is the representation of anchored point comparability as those signed constraints, together with the resulting interpretation and exact accord count. For positive pair labels without a common anchor, [19] uses the same parity constraints to construct a closure predictor and a sample compression scheme.

4.4. **A canonical bipartite form and rank-one completion.** The constraint graph of Theorem 4.3 depends on the chosen roots. There is a root-free companion object in which the auxiliary signs τ_x of Theorem 3.2 are retained rather than eliminated.

Definition 4.10 (Bipartite incidence graph)**.** For $A \subseteq D \setminus \{q\}$, the signed bipartite incidence graph $B_q(A)$ has vertex set $A \sqcup [d]$ and, for each $x \in A$ and each $i \in S_q(x)$, one edge xi with sign $\alpha_q(x)_i$.

Proposition 4.11 (Canonical bipartite form)**.** *Let $A \subseteq D \setminus \{q\}$ and let e_0 be the number of $x \in A$ with $x \equiv q$.*

(a) A pair $(\tau, \sigma) \in \{-,+\}^A \times \Sigma_d$ is a satisfying vertex signing of $B_q(A)$, meaning $\tau_x \sigma_i = \alpha_q(x)_i$ for every edge xi, if and only if $\sigma \in \mathrm{Acc}_q(A)$ and, for every x with $S_q(x) \neq \varnothing$, the sign τ_x is the direction of comparison of Theorem 3.2.

(b) A is compatible about q if and only if $B_q(A)$ is balanced.

(c) $B_q(A)$ has exactly $c_q(A) + e_0$ connected components, and when balanced it has $2^{c_q(A)+e_0}$ satisfying signings, projecting 2^{e_0}-to-one onto $\mathrm{Acc}_q(A)$.

Proof. (a) is the first formulation of Theorem 3.2: given σ, the equations $\tau_x \sigma_i = \alpha_q(x)_i$, $i \in S_q(x)$, are solvable for τ_x exactly when $\sigma \in C_q(x)$, and the solution is unique when $S_q(x) \neq \varnothing$; points with $x \equiv q$ are isolated and leave τ_x free.

(b) A signed graph is balanced if and only if it admits a satisfying vertex signing [14]; combine with (a).

(c) Two coordinate vertices lie in one component of $B_q(A)$ exactly when they are linked by a chain of overlapping supports, which is the component relation of $G_q(A)$; every point vertex with nonempty support is attached to its support, and the e_0 points with empty support are isolated. The count of signings and the fibres of the projection follow as in Theorem 4.5. □

Remark 4.12 (Rank-one sign completion)*.* Record the anchored data as a partial sign matrix $M \in \{-,+,*\}^{A \times [d]}$ with $M_{x,i} = \alpha_q(x)_i$ for $i \in S_q(x)$ and $M_{x,i} = *$ otherwise. By Theorem 4.11, A is compatible about q if and only if M admits a completion of the form $M_{x,i} = \tau_x \sigma_i$, that is, a completion to a real matrix of rank one with ± 1 entries. The completions correspond to satisfying signings up to global negation, so a compatible family has exactly $2^{c_q(A)+e_0-1}$ of them. In signed-graph language, (τ, σ) is a switching of $B_q(A)$ onto the all-positive graph. Anchored compatibility is therefore a rank-one sign-completion problem, and Theorem 4.5 counts its solutions.

5. Obstructions and the compatibility complex

5.1. **Negative cycles are the obstructions.** Theorem 4.5 identifies every failure of compatibility with a negative cycle. Importantly, such a cycle need not be a triangle.

Theorem 5.1 (Unbounded minimal obstructions)**.** *For every integer $k \geq 3$, there exist an anchor $q \in \mathbb{R}^k$ and a set $A = \{x_1, \ldots, x_k\}$ such that A is not compatible about q, while every proper subset of A is compatible.*

Proof. Let $q = 0$, and let $e_1, \ldots, e_k$ be the standard basis of $\mathbb{R}^k$. Set

$$x_i = e_i + e_{i+1} \quad (1 \leq i < k), \qquad x_k = e_k - e_1.$$

For $i < k$, the point x_i imposes $\sigma_i \sigma_{i+1} = +$. The last point imposes $\sigma_k \sigma_1 = -$. Thus the constraint graph is a k-cycle with exactly one negative edge, and is not balanced.

Deleting any point deletes one edge from this cycle. The remaining constraint graph is a forest and hence balanced. The claim follows from Theorem 4.5. □

Corollary 5.2 (No dimension-free Helly number)**.** *There is no constant h such that a finite point set is compatible about an anchor whenever all of its subsets of size at most h are compatible.*

Proof. Choose $k > h$ in Theorem 5.1. □

For fixed dimension, however, the Helly number is finite and can be determined exactly.

Theorem 5.3 (Helly number equals the dimension)**.** *Let $d \geq 2$. Every family $A \subseteq D \setminus \{q\}$ that is not compatible about q contains an incompatible subfamily of at most d points. Hence A is compatible whenever all of its subfamilies of size at most d are compatible, and by Theorem 5.1 the bound d is attained for every $d \geq 3$; for $d = 2$ it is attained by a pair of points imposing conflicting parities, such as $x = (1, 1)$ and $y = (1, -1)$ about $q = (0, 0)$.*

Proof. The sign of a cycle is determined by the parity of its negative edges, which is additive under symmetric difference. Since the cycle space of a multigraph is spanned by its simple cycles, a signed multigraph all of whose simple cycles are positive is balanced; equivalently, an unbalanced signed multigraph contains a simple negative cycle. A simple cycle in a multigraph on $d \geq 2$ vertices has at most d edges, the case of two parallel edges included.

Suppose A is incompatible. By Theorem 4.5, $G_q(A)$ is unbalanced, so it contains a simple negative cycle Z with at most d edges. For each edge of Z, select one point of A generating it, and let A' be the set of selected points, so $|A'| \leq d$. Constructed with the same roots, $G_q(A')$ contains Z and is therefore unbalanced, so A' is incompatible by Theorem 4.5. □

Corollary 5.4 (Helly number two in general position)**.** *Suppose $x_i \neq q_i$ for every $x \in A$ and every $i \in [d]$. If every pair of points of A is compatible about q, then A is compatible about q. Consequently, minimal incompatible families with more than two points must contain points agreeing with the anchor in some coordinate.*

Proof. By Theorem 4.7, two full-support points are compatible about q exactly when their sign vectors $\chi(q, \cdot)$ agree up to antipodes. Agreement up to antipodes is an equivalence relation, so pairwise compatibility places all sign vectors in a single antipodal class, and Theorem 4.7 applies to the whole family. □

The Helly number of anchored compatibility is therefore dimension-bounded but not dimension-free, and coordinates tied to the anchor are the only source of minimal obstructions with more than two points. Theorem 5.5 below realises the bound of Theorem 5.3 in dimension four.

Example 5.5 (A four-point obstruction)**.** In $\mathbb{R}^4$, take

$$q = (0,0,0,0)$$

and

$$x_1 = (1,1,0,0), \quad x_2 = (0,1,1,0), \quad x_3 = (0,0,1,1), \quad x_4 = (1,0,0,-1).$$

The constraints are

$$\sigma_1 = \sigma_2, \quad \sigma_2 = \sigma_3, \quad \sigma_3 = \sigma_4, \quad \sigma_1 \neq \sigma_4.$$

Every triple has two compatible orientations, but the four-point set has none.

5.2. **The compatibility complex.**

Definition 5.6 (Anchored compatibility complex)**.** For $q \in D$, define

$$\mathcal{K}_q(D) = \{A \subseteq D \setminus \{q\} : \mathrm{Acc}_q(A) \neq \varnothing\}.$$

Since compatibility is inherited by subsets, $\mathcal{K}_q(D)$ is an abstract simplicial complex. Its minimal nonfaces are precisely the point-minimal families whose signed constraint graphs contain a negative cycle.

There is a second, geometric description. Let

$$\mathbb{P}\Sigma_d = \Sigma_d/\{\sigma \sim -\sigma\}$$

be the 2^{d-1}-element set of antipodal orientation pairs. For $[\sigma] \in \mathbb{P}\Sigma_d$, put

$$B_q([\sigma]) = \{x \in D \setminus \{q\} : \sigma \in C_q(x)\}.$$

This is well-defined because $C_q(x)$ is antipodally invariant.

Proposition 5.7 (Simplex decomposition)**.** *The compatibility complex satisfies*

$$\mathcal{K}_q(D) = \bigcup_{[\sigma] \in \mathbb{P}\Sigma_d} 2^{B_q([\sigma])},$$

where 2^B denotes the full simplex on vertex set B. Its facets are the inclusion-maximal distinct sets among the $B_q([\sigma])$.

Proof. A subset A lies in $\mathcal{K}_q(D)$ if and only if some orientation σ lies in every $C_q(x)$, $x \in A$. This is equivalent to $A \subseteq B_q([\sigma])$. The description of the facets follows. □

Proposition 5.8 (Facets and the general-position complex)**.** *The complex $\mathcal{K}_q(D)$ has at most 2^{d-1} facets, and for every m with $1 \leq m \leq 2^{d-1}$ there are D and q attaining exactly m facets. If moreover $x_i \neq q_i$ for every $x \in D \setminus \{q\}$ and $i \in [d]$, then the nonempty sets $B_q([\sigma])$ partition $D \setminus \{q\}$ into antipodal sign classes, and $\mathcal{K}_q(D)$ is a disjoint union of at most 2^{d-1} full simplices; in particular it is homotopy equivalent to a set of at most 2^{d-1} points.*

Proof. By Theorem 5.7, every facet is one of the at most 2^{d-1} sets $B_q([\sigma])$. For attainment, take $q = 0$, choose m distinct classes $[\sigma_1], \ldots, [\sigma_m]$, and let x_j be the ± 1-vector σ_j, so that $\chi(q, x_j) = \sigma_j$ has full support. A full-support point is comparable with q only in the antipodal class of its sign vector.

Hence $B_q([\sigma_j]) = \{x_j\}$, while every unchosen class has an empty point set. Since $m \geq 1$, the empty set is not inclusion-maximal, and the complex has exactly the m singleton facets.

In general position every x has full support, so by Theorem 2.6 it is comparable with q exactly in the orientations of the single class $[\chi(q, x)]$. Hence each point lies in exactly one set $B_q([\sigma])$, the nonempty sets partition $D \setminus \{q\}$, and the simplices of Theorem 5.7 are pairwise disjoint. A disjoint union of k full simplices deformation retracts onto k points. □

Thus ties with the anchor are what entangle the facets: without them the compatibility complex is homotopically discrete, in accordance with the Helly number two of Theorem 5.4.

Theorem 5.7 places $\mathcal{K}_q(D)$ between two familiar combinatorial objects: it is generated by antipodal Boolean orientations, while its missing faces are detected by negative cycles in signed coordinate graphs. We now show that this decomposition controls the topology of the complex completely.

5.3. **Dowker duality and the homotopy type.** Anchored comparability is a binary relation

$$R_q = \{(x, [\sigma]) : \sigma \in C_q(x)\} \subseteq (D \setminus \{q\}) \times \mathbb{P}\Sigma_d,$$

and $\mathcal{K}_q(D)$ is its Dowker complex on the point side [10]. The Dowker complex on the orientation side is the following nerve.

Definition 5.9 (Orientation nerve)**.** Let $\nu_q(D)$ be the simplicial complex whose vertices are the classes $[\sigma] \in \mathbb{P}\Sigma_d$ with $B_q([\sigma]) \neq \varnothing$, a set T of classes forming a face precisely when $\bigcap_{[\sigma]\in T} B_q([\sigma]) \neq \varnothing$, that is, when some point is comparable with q in every orientation of T.

Theorem 5.10 (Dowker duality)**.** *The geometric realisations of $\mathcal{K}_q(D)$ and $\nu_q(D)$ are homotopy equivalent.*

Proof. By Theorem 5.7, the closed simplices $2^{B_q([\sigma])}$ cover $\mathcal{K}_q(D)$. For every set T of classes,

$$\bigcap_{[\sigma]\in T} 2^{B_q([\sigma])} = 2^{\bigcap_{[\sigma]\in T} B_q([\sigma])},$$

which is either empty or a full simplex, hence contractible. The nerve of this cover is $\nu_q(D)$ by definition, so the nerve lemma for covers of polyhedra by subcomplexes with contractible nonempty intersections [5, Theorem 10.6] yields the homotopy equivalence. Equivalently, the statement is Dowker's theorem [10] applied to the relation R_q. □

Corollary 5.11 (The homotopy type depends on d only)**.** *The homotopy type of $\mathcal{K}_q(D)$ is that of a simplicial complex on at most 2^{d-1} vertices. Consequently, all reduced Betti numbers of $\mathcal{K}_q(D)$ vanish in dimensions above $2^{d-1} - 2$, they are bounded by a function of d alone, and for fixed d only*

finitely many homotopy types occur as D and q vary, independently of $|D|$. Moreover,

$$\chi(\mathcal{K}_q(D)) = \sum_{\substack{\varnothing \neq T \subseteq \mathbb{P}\Sigma_d \\ \bigcap_{[\sigma]\in T} B_q([\sigma]) \neq \varnothing}} (-1)^{|T|+1},$$

whenever $\mathcal{K}_q(D)$ is nonempty.

Proof. Put $M = 2^{d-1}$. The complex $\nu_q(D)$ has at most M vertices. If its dimension is $M-1$, then it contains the face consisting of all M vertices and is therefore the full simplex, hence contractible; otherwise its dimension is at most $M-2$. Thus its reduced homology vanishes above dimension $M-2$. There are finitely many complexes on a fixed vertex set. Apply Theorem 5.10. The Euler characteristic formula is inclusion–exclusion over the cover by full simplices, each nonempty intersection contributing $\chi = 1$. □

Remark 5.12 (Points and orientations exchange roles). Under the duality, the vertex x of $\mathcal{K}_q(D)$ corresponds in $\nu_q(D)$ to the simplex on $\{[\sigma] : \sigma \in C_q(x)\}$, which is the image in the folded cube $\mathbb{P}\Sigma_d$ of a Boolean face of dimension $e(q, x)$. Thus $\mathcal{K}_q(D)$ is a union of simplices indexed by orientation classes, while $\nu_q(D)$ is a union of simplices on subcubes indexed by points: the compatibility complex is, up to homotopy, the nerve of a family of subcubes of the folded cube. Large data sets can therefore enrich the face structure of $\mathcal{K}_q(D)$ without ever leaving a finite list of homotopy types.

5.4. **The largest compatible subset.** Recognition of compatibility is linear-time (Theorem 9.1), and Theorem 4.5 even counts the compatible orientations. The associated extremal problem is, by contrast, hard.

Theorem 5.13 (Hardness of extremal accord)**.** *Given q and D, deciding whether some compatible $A \subseteq D \setminus \{q\}$ has $|A| \geq k$ is NP-complete, already for $q = 0$, points in $\{-1, 0, 1\}^d$, and $|S_q(x)| = 2$ for every point. Hence computing the maximum size of a compatible subset is NP-hard.*

Proof. Membership in NP: guess A and an orientation σ, and verify $\sigma \in \mathrm{Acc}_q(A)$ in linear time.

For hardness, reduce from Simple Max Cut, which is NP-complete [11, Problem ND16]; see also [12]. The maximum size of a bipartite edge subset equals the maximum cut size: the crossing edges of any cut are bipartite, while a bipartition of any bipartite edge subset extends to a cut containing that subset. Given a simple graph $H = ([d], E)$, set $q = 0$ and, for each edge $uv \in E$, create the point $x_{uv} = e_u - e_v \in \{-1, 0, 1\}^d$. Its constraint is $\sigma_u \sigma_v = \alpha_q(x_{uv})_u \alpha_q(x_{uv})_v = -$, so a subset $A \subseteq \{x_{uv} : uv \in E\}$ is compatible about q exactly when the corresponding all-negative signed subgraph is balanced, that is, when the corresponding edge set is bipartite. Since each point contributes a single edge, maximum compatible subsets correspond to maximum bipartite edge subgraphs of H. □

Remark 5.14. The reduction identifies extremal accord with the maximum balanced subgraph problem for signed graphs; in particular the all-negative case is exactly max cut, so semidefinite approximation guarantees [13] and the corresponding hardness of approximation transfer to the point-deletion problem with binary supports. Points with larger supports bundle several edges into one deletable unit, a labelled variant that appears again in Section 11 below.

6. Incomparability and antichain orientations

The failure of anchored compatibility is not itself a simultaneous incomparability statement. Indeed, $\mathrm{Acc}_q(A) = \varnothing$ says that every orientation leaves *some* point incomparable with q; it does not say that one orientation leaves *all* points incomparable with q. We now study the latter requirement and then remove the anchor altogether.

6.1. **Anchored anti-accord.**

Definition 6.1 (Anchored anti-accord)**.** For $A \subseteq D \setminus \{q\}$, define

$$\mathrm{Inc}_q(A) = \{\sigma \in \Sigma_d : q \not\preceq_\sigma x \text{ and } x \not\preceq_\sigma q \text{ for every } x \in A\}.$$

Thus

$$\mathrm{Inc}_q(A) = \bigcap_{x \in A} (\Sigma_d \setminus C_q(x)) = \Sigma_d \setminus \bigcup_{x \in A} C_q(x).$$

For example, take $q = (0,0)$, $x = (1,1)$, and $y = (1,-1)$. Then $C_q(x) = \{++, --\}$ and $C_q(y) = \{+-, -+\}$. Hence $\mathrm{Acc}_q(\{x,y\}) = \varnothing$, but also $\mathrm{Inc}_q(\{x,y\}) = \varnothing$: in every orientation exactly one of the two points is comparable with the anchor.

Lemma 6.2 (Signed not-all-equal representation)**.** *An orientation σ lies in $\mathrm{Inc}_q(A)$ if and only if, for every $x \in A$,*

$$\{\sigma_i \alpha_q(x)_i : i \in S_q(x)\} = \{-, +\}.$$

Equivalently, after the binary identification of signs, the variables

$$b_i + \beta_q(x)_i, \qquad i \in S_q(x),$$

are not all equal.

Proof. By Theorem 3.2, q and x are comparable in orientation σ exactly when $\sigma_i \alpha_q(x)_i$ is constant on $S_q(x)$. Negating this condition gives the claim. When $|S_q(x)| \le 1$, the displayed set cannot contain both signs, consistently with the fact that such a point is comparable with q in every orientation. □

Thus anchored compatibility is an equality or parity problem, whereas anchored incomparability is a signed not-all-equal (NAE) problem. In Boolean cube language, $\mathrm{Inc}_q(A) \neq \varnothing$ precisely when the pairs of antipodal faces $C_q(x)$, $x \in A$, do not cover Σ_d. Coverings by antipodal face pairs also occur in hypergraph DP-colouring [4, 20]; the present constraints are induced

by anchored coordinate comparisons, and both their intersection and their avoidance are retained.

Theorem 6.3 (Affine–NAE complexity boundary)**.** *Consider the decision problem whether* $\mathrm{Inc}_q(A)$ *is nonempty, with the support–sign pairs* $(S_q(x), \alpha_q(x))$ *given as input.*

(a) If $|S_q(x)| \le 2$ *for every* $x \in A$*, recognition and counting take* $O(d+|A|)$ *time. The anti-accord is empty or an affine subspace of* $\mathbb{F}_2^d$*. More precisely, a support of size at most one makes it empty; otherwise, put an edge* ij *of sign* $-\alpha_q(x)_i\alpha_q(x)_j$ *for each* $S_q(x) = \{i, j\}$*. If this signed graph is balanced with* c *connected components, then* $|\mathrm{Inc}_q(A)| = 2^c$*, and otherwise it is empty.*

(b) The decision problem is NP-complete even when $q = 0$*, every point belongs to* $\{-1, 0, 1\}^d$*, and* $|S_q(x)| = 3$ *for every* $x \in A$*.*

Proof. For (a), a point with support of size zero or one makes the anti-accord empty. Otherwise every support has size two, say $S_q(x) = \{i, j\}$. By Theorem 6.2, its constraint is

$$\sigma_i\sigma_j = -\alpha_q(x)_i\alpha_q(x)_j.$$

Build the signed multigraph on $[d]$ having one edge ij with this sign for every x. The proof of Theorem 4.5 applies verbatim: the system is feasible exactly when the graph is balanced, and if it has c connected components then it has 2^c solutions. Construction and signed graph search take $O(d + |A|)$ time.

For (b), membership in NP is immediate. We reduce signed NAE-3-SAT, which is NP-complete [21]. Let its Boolean variables be represented by signs $\sigma_1, \dots, \sigma_d$. Write a clause on distinct variables as the requirement that

$$\gamma_i\sigma_i, \quad \gamma_j\sigma_j, \quad \gamma_k\sigma_k$$

are not all equal, where each $\gamma \in \{-, +\}$ records whether the corresponding literal is negated. With $q = 0$, create the point x whose coordinates on $\{i, j, k\}$ are $\gamma_i, \gamma_j, \gamma_k$ and whose other coordinates are zero. Then $S_q(x) = \{i, j, k\}$ and $\alpha_q(x)_r = \gamma_r$ on its support. By Theorem 6.2, incomparability with this point is exactly the given clause. The construction is polynomial and uses only $\{-1, 0, 1\}$-valued points. □

Theorem 6.3 gives a sharp change within the orthant framework: binary supports produce the same affine geometry as signed balance, while ternary supports already express general NAE constraints.

Although the anti-accord is no longer affine, its cardinality is still governed by the accord numbers of subfamilies.

Proposition 6.4 (Anti-accord by inclusion–exclusion)**.** *For every finite* $A \subseteq D \setminus \{q\}$*,*

$$|\mathrm{Inc}_q(A)| = \sum_{B \subseteq A} (-1)^{|B|}\, a_q(B) = \sum_{\substack{B \subseteq A \\ B \text{ compatible about } q}} (-1)^{|B|}\, 2^{c_q(B)},$$

with the conventions $a_q(\varnothing) = 2^d$ *and* $c_q(\varnothing) = d$.

Proof. Since $\operatorname{Inc}_q(A) = \Sigma_d \setminus \bigcup_{x\in A} C_q(x)$, inclusion–exclusion gives

$$|\operatorname{Inc}_q(A)| = \sum_{B\subseteq A} (-1)^{|B|} \left| \bigcap_{x\in B} C_q(x) \right|.$$

The intersection is $\operatorname{Acc}_q(B)$. Incompatible subfamilies contribute zero, and compatible ones contribute $2^{c_q(B)}$ by Theorem 4.5. □

Combined with Theorem 9.1, the formula evaluates $|\operatorname{Inc}_q(A)|$ in $O\big(2^{|A|}(d + L_q(A))\big)$ time, exponential only in the number of points. Dependence on $2^{|A|}$ cannot be removed in general:

Corollary 6.5 (Counting the anti-accord is #P-complete)**.** *Computing* $|\operatorname{Inc}_q(A)|$ *is #P-complete under polynomial-time Turing reductions when supports of size three are allowed, already for* $q = 0$ *and points in* $\{-1, 0, 1\}^d$. *For supports of size at most two it is computed in* $O(d + |A|)$ *time by Theorem 6.3.*

Proof. Membership in #P is immediate. By the Boolean counting dichotomy, counting the satisfying assignments of signed NAE-3-SAT is #P-complete under polynomial-time Turing reductions because the not-all-equal relations are not affine [9]. The reduction in the proof of Theorem 6.3(b) is parsimonious: its coordinates are precisely the source variables, its points are precisely the clauses, and orientations in $\operatorname{Inc}_q(A)$ correspond bijectively to satisfying assignments. A variable occurring in no clause becomes an isolated coordinate and remains free on both sides of the bijection. □

Compatibility and incomparability requirements may also be mixed, and the two theorems above are then the faces of a single dichotomy.

Definition 6.6 (Mixed accord)**.** For disjoint $A_+, A_- \subseteq D \setminus \{q\}$, let

$$\operatorname{Acc}_q(A_+, A_-) = \operatorname{Acc}_q(A_+) \cap \operatorname{Inc}_q(A_-),$$

the set of orientations comparing every point of A_+ with q and leaving every point of A_- incomparable with q.

Theorem 6.7 (Mixed dichotomy)**.** *Consider the decision problem whether* $\operatorname{Acc}_q(A_+, A_-)$ *is nonempty, with the support–sign pairs given as input and with the supports on the incomparability side bounded by a constant* s*; the supports on the comparability side are unrestricted.*

(a) If $s \le 2$, *then* $\operatorname{Acc}_q(A_+, A_-)$ *is empty or a coset of a subspace of* $\mathbb{F}_2^d$ *containing* $\mathbf{1}$, *recognition and counting take* $O\big(d + \sum_{x\in A_+} |S_q(x)| + |A_-|\big)$ *time, and the count is* 2^c *with* c *the number of components of a signed multigraph combining both constraint types.*

(b) If $s \ge 3$, *the decision problem is NP-complete, already with* $A_+ = \varnothing$, $q = 0$, *and points in* $\{-1, 0, 1\}^d$.

Proof. For (a), a point of A_- with $|S_q(x)| \le 1$ makes the set empty. Otherwise, place on $[d]$ the star edges $\sigma_i\sigma_j = \alpha_q(x)_i\alpha_q(x)_j$ for $x \in A_+$, as in Theorem 4.3, and one edge $\sigma_i\sigma_j = -\alpha_q(x)_i\alpha_q(x)_j$ for each $x \in A_-$ with $S_q(x) = \{i, j\}$, as in Theorem 6.3(a). All constraints are parity equations on pairs of variables, so the solution set is empty or a coset, invariant under $\mathbf{1}$; the proof of Theorem 4.5 applies verbatim to the combined signed multigraph and gives feasibility, certificates, and the count 2^c in the stated time.

Part (b) contains Theorem 6.3(b) as the special case $A_+ = \varnothing$, and membership in NP is again immediate. □

Thus arbitrarily large comparability supports never leave the affine regime, whereas a single unit of slack beyond binary incomparability supports is already universal for NAE constraints. In the affine regime the Helly bound of Theorem 5.3 also persists: an infeasible mixed system with $s \le 2$ contains an infeasible subfamily of at most d points, namely a single point of A_- with support of size at most one, or the points generating a simple negative cycle of the combined signed multigraph.

6.2. **Pairwise antichain orientations.**

Definition 6.8 (Antichain orientation set)**.** For $A \subseteq D$, let

$$\mathrm{Ant}(A) = \{\sigma \in \Sigma_d : \text{distinct } x, y \in A \text{ are incomparable under } \preceq_\sigma\}.$$

Equivalently,

$$\mathrm{Ant}(A) = \Sigma_d \setminus \bigcup_{\{x,y\} \in \binom{A}{2}} C(x, y).$$

Unlike $\mathrm{Inc}_q(A)$, which uses one constraint per point relative to an anchor, $\mathrm{Ant}(A)$ uses the sign patterns of all point pairs. The pair count already gives a useful global bound.

Theorem 6.9 (Antichain orientation bound)**.** *For every finite $A \subseteq D$,*

$$|\mathrm{Ant}(A)| \ge 2^d - \sum_{\{x,y\} \in \binom{A}{2}} 2^{\min\{e(x,y)+1,\, d\}}.$$

In particular, if the sum on the right is smaller than 2^d, then some orientation makes A an antichain.

Proof. The complement of $\mathrm{Ant}(A)$ is the union of the pairwise comparability sets $C(x, y)$. Apply the union bound and then Theorem 2.7. □

Corollary 6.10 (General-position antichain orientation)**.** *Suppose no two distinct points of an N-point set A agree in any coordinate. Then*

$$|\mathrm{Ant}(A)| \ge 2^d - 2\binom{N}{2}.$$

Consequently,

$$\binom{N}{2} < 2^{d-1}$$

guarantees an orientation under which A is an antichain.

The estimate is deliberately distribution-free. Overlaps among the antipodal pairs $C(x, y)$ can only increase $|\operatorname{Ant}(A)|$, and their geometry contains information not captured by the union bound.

6.3. **Orientation density and a high-dimensional limit.** The exact counts admit a probabilistic reading even for deterministic data. If an orientation is chosen uniformly from Σ_d, define

$$\rho(x, y) = \frac{|C(x, y)|}{2^d}.$$

For $x \not\equiv y$, Theorem 2.7 gives

$$\rho(x, y) = 2^{1-|S(x,y)|}, \qquad S(x, y) = \{i : x_i \neq y_i\};$$

for duplicate coordinate vectors, $\rho(x, y) = 1$. Similarly, the anchored orientation density is

$$\frac{|\operatorname{Acc}_q(A)|}{2^d} = \begin{cases} 2^{c_q(A)-d}, & A \text{ is compatible about } q, \\ 0, & A \text{ is incompatible about } q. \end{cases}$$

Thus $d - c_q(A)$ is the number of independent binary constraints imposed on a compatible orientation family.

Theorem 6.11 (Fixed-orientation antichain limit)**.** *Let $X^{(1)}, \dots, X^{(N)}$ be independent random points in $\mathbb{R}^d$. Assume that all coordinates are independent and that, in each coordinate, the N values are identically distributed with a continuous law. For any fixed $\sigma \in \Sigma_d$, let K_σ be the number of comparable unordered pairs under $\preceq_\sigma$. Then*

$$\mathbb{E}K_\sigma = \binom{N}{2} 2^{1-d}$$

and

$$\Pr(K_\sigma > 0) \leq \binom{N}{2} 2^{1-d}.$$

Consequently, if $N^2 2^{-d} \to 0$, then the sample is an antichain under the fixed orientation with probability tending to one.

Proof. For a fixed pair, continuity excludes ties. In each coordinate either point is smaller with probability $1/2$, independently across coordinates. Comparability requires all d comparisons to agree with σ, or all to disagree with it, and hence has probability 2^{1-d}. Linearity of expectation gives the first formula, and Markov's inequality gives the second. □

The scale in Theorem 6.11 is $d \approx 2 \log_2 N$. More precise thresholds for chains in random high-dimensional product orders were obtained by Bollobás and Brightwell [7]; the two-point case already explains why a fixed product order becomes almost an antichain in high-dimensional data. At the threshold scale itself, the first-moment bound sharpens to a limit law.

Theorem 6.12 (Poisson limit at the antichain threshold)**.** *In the setting of Theorem 6.11, suppose that, as $N \to \infty$, $d = d(N) \to \infty$ and*

$$\lambda_N = \binom{N}{2} 2^{1-d} \longrightarrow \lambda \in [0, \infty).$$

Then K_σ converges in distribution to a Poisson random variable with mean λ; in particular,

$$\Pr(K_\sigma = 0) \longrightarrow e^{-\lambda}.$$

Consequently, along every sequence satisfying $d - 2\log_2 N \to t \in \mathbb{R}$, the probability that the sample is an antichain under $\preceq_\sigma$ converges to $\exp(-2^{-t})$.

Proof. Index the indicators by unordered pairs: for $\pi = \{u, v\}$, let I_π be the event that $X^{(u)}$ and $X^{(v)}$ are comparable under $\preceq_\sigma$, so $p_\pi = \mathbb{E}I_\pi = 2^{1-d}$ and $K_\sigma = \sum_\pi I_\pi$. Take $B_\pi = \{\pi' : \pi' \cap \pi \neq \varnothing\}$, including π itself, as the dependence neighbourhood. The indicator I_π is measurable with respect to the two points indexed by π, whereas the entire collection $\{I_{\pi'} : \pi' \notin B_\pi\}$ is measurable with respect to the remaining points. The two sigma-fields are independent, so the Chen–Stein term b_3 vanishes.

The first term obeys

$$b_1 = \sum_{\pi} \sum_{\pi' \ni \text{ a point of } \pi} p_\pi p_{\pi'} = \binom{N}{2} (2N - 3) \left(2^{1-d}\right)^2 = O(N^3 4^{-d}).$$

For the second, fix distinct pairs $\{u, v\}$ and $\{u, w\}$, and write $x = X^{(u)}$, $y = X^{(v)}$, $z = X^{(w)}$, $s_i = \operatorname{sgn}(y_i - x_i)$, and $t_i = \operatorname{sgn}(z_i - x_i)$. In each coordinate, the six equiprobable strict rankings of the three independent continuous values give $\Pr(s_i = a, t_i = b) = \frac{1}{3}$ if $a = b$ and $\frac{1}{6}$ if $a \neq b$. Both pairs are comparable exactly when $s = \tau\sigma$ and $t = \tau'\sigma$ for some $\tau, \tau' \in \{-, +\}$, whence

$$\Pr(I_\pi = I_{\pi'} = 1) = 2 \cdot 3^{-d} + 2 \cdot 6^{-d},$$

and

$$b_2 = \binom{N}{2} 2(N - 2)(2 \cdot 3^{-d} + 2 \cdot 6^{-d}) = O(N^3 3^{-d}).$$

Since $N^2 2^{-d}$ is bounded, $N^3 4^{-d} = (N^2 2^{-d})^{3/2} 2^{-d/2} \to 0$ and $N^3 3^{-d} = (N^2 2^{-d})^{3/2} (2^{3/2}/3)^d \to 0$, using $2^{3/2} < 3$. By the Chen–Stein theorem [3, Theorem 1], the total variation distance between the law of K_σ and the Poisson law with mean λ_N is at most $2(b_1 + b_2) \to 0$, and $\lambda_N \to \lambda$. If $d - 2\log_2 N \to t$, then

$$\lambda_N = (1 - N^{-1}) 2^{-(d - 2\log_2 N)} \longrightarrow 2^{-t}.$$

□

The property of being a fixed-orientation antichain therefore has a coarse threshold at $d = 2\log_2 N$ with the explicit profile $t \mapsto \exp(-2^{-t})$: two extra

dimensions per doubling of the sample, with the transition spread over an additive window in d.

Corollary 6.13 (Expected number of antichain orientations)**.** *In the setting of Theorem 6.12, let $A = \{X^{(1)}, \dots, X^{(N)}\}$. Then*

$$\mathbb{E}\,|\operatorname{Ant}(A)| = (e^{-\lambda} + o(1))2^d.$$

In particular, if $d - 2\log_2 N \to t$, then the expected proportion of orientations under which the sample is an antichain converges to $\exp(-2^{-t})$.

Proof. We have $\mathbb{E}\,|\operatorname{Ant}(A)| = \sum_{\sigma\in\Sigma_d} \Pr(K_\sigma = 0)$. The probabilities entering the proof of Theorem 6.12 do not depend on σ: a fixed pair is comparable with probability 2^{1-d}, and two pairs sharing a point are both comparable with probability $2\cdot 3^{-d} + 2\cdot 6^{-d}$. Hence the Chen–Stein bound holds uniformly:

$$\sup_{\sigma\in\Sigma_d} \left|\Pr(K_\sigma = 0) - e^{-\lambda_N}\right| \le 2(b_1 + b_2) = o(1).$$

Consequently,

$$\left|2^{-d}\mathbb{E}|\operatorname{Ant}(A)| - e^{-\lambda_N}\right| \le \sup_{\sigma\in\Sigma_d} \left|\Pr(K_\sigma = 0) - e^{-\lambda_N}\right| = o(1).$$

Since $\lambda_N \to \lambda$, the claim follows. □

Proposition 6.14 (Random anchored compatibility)**.** *Let $q = 0$, let $m \ge 1$, and let $X^{(1)}, \dots, X^{(m)}$ be independent random points whose coordinates are independent, continuous, and symmetric about zero. Then*

$$\Pr(\{X^{(1)}, \dots, X^{(m)}\} \text{ is compatible about } q) = 2^{-(d-1)(m-1)}.$$

Proof. Each sign vector $\chi(q, X^{(j)})$ is uniform on Σ_d, and the vectors are independent. By Theorem 4.7, compatibility means that all m vectors lie in the same antipodal class. After the first class is fixed, each remaining point matches it with probability $2/2^d = 2^{1-d}$. □

7. Intrinsic incomparability beyond chains

The chain assumption isolates incomparability caused by conflicting coordinate directions: every coordinate pair is itself comparable, and a suitable orientation compares every point pair. Nominal, hierarchical, and other structured data naturally lead instead to products of posets. In that setting there is an additional, orientation-independent obstruction.

Let $P_1, \dots, P_d$ be posets and let D map into their product. As before, an orientation σ selects either P_i or its dual in each coordinate. Define the extended sign map

$$\widehat{\chi}(x,y)_i = \begin{cases} +, & x_i < y_i, \\ 0, & x_i = y_i, \\ -, & y_i < x_i, \\ \|, & x_i \text{ and } y_i \text{ are incomparable in } P_i. \end{cases}$$

Write

$$J(x,y) = \{i : \widehat{\chi}(x,y)_i = \|\}.$$

Theorem 7.1 (Intrinsic and orientation-induced incomparability)**.** *For a point pair in a product of posets, let*

$$\widehat{C}(x,y) = \{\sigma \in \Sigma_d : x \preceq_\sigma y \text{ or } y \preceq_\sigma x\}.$$

If $J(x,y) \neq \varnothing$*, then* $\widehat{C}(x,y) = \varnothing$*. If* $J(x,y) = \varnothing$*, then* $\widehat{\chi}(x,y) \in \{-,0,+\}^d$ *and*

$$\widehat{C}(x,y) = F(\widehat{\chi}(x,y)) \cup F(-\widehat{\chi}(x,y)).$$

Consequently,

$$\bigcup_{\sigma\in\Sigma_d} \preceq_\sigma = \{(x,y) \in D \times D : x_i \text{ and } y_i \text{ are comparable in } P_i \text{ for every } i\}.$$

Proof. Dualising a poset reverses strict comparisons but does not make an incomparable pair comparable. Thus one coordinate in $J(x,y)$ prevents a product comparison in every orientation. When $J(x,y)$ is empty, every coordinate relation is $<$, $=$, or $>$, and the proof of Theorems 2.4 and 2.6 applies coordinate by coordinate. Finally, if all coordinate pairs are comparable, orient each coordinate so that x_i is below y_i; the resulting product orientation contains (x,y). □

A nominal coordinate has a canonical flat-poset model:

$$a \leq b \quad \Longleftrightarrow \quad a = b.$$

Distinct strings or labels are then intrinsically incomparable, independently of coordinate reversal. A taxonomy, prefix relation, or set-inclusion relation may instead supply a nontrivial application-specific poset. Under the strict product rule of Theorem 7.1, an intrinsically incomparable coordinate is a hard veto for compatibility and an automatic witness for incomparability. This is mathematically distinct from the signed NAE obstruction of Theorem 6.2, which arises even though every coordinate pair lies in a chain.

8. Symmetries, covariance, and projections

8.1. **Coordinatewise monotone transformations.** Assume for this subsection that the coordinate chains are real intervals. Let

$$f(x_1,\dots,x_d) = (f_1(x_1),\dots,f_d(x_d)),$$

where each f_i is strictly monotone. Define $\varepsilon(f) \in \Sigma_d$ by

$$\varepsilon(f)_i = \begin{cases} +, & f_i \text{ is increasing,} \\ -, & f_i \text{ is decreasing.} \end{cases}$$

Theorem 8.1 (Order covariance)**.** *For all* $x, y \in D$*,*

$$\chi(f(x), f(y)) = \varepsilon(f)\chi(x,y),$$

where zero coordinates remain zero. Consequently,

$$x \preceq_\sigma y \quad \Longleftrightarrow \quad f(x) \preceq_{\varepsilon(f)\sigma} f(y).$$

For every anchor q and $A \subseteq D \setminus \{q\}$,

$$\mathrm{Acc}_{f(q)}(f(A)) = \varepsilon(f)\,\mathrm{Acc}_q(A).$$

In particular, compatibility and accord numbers are invariant under coordinatewise strictly monotone transformations.

Proof. A strictly increasing coordinate map preserves $<, =, >$, whereas a strictly decreasing one interchanges $<$ and $>$ and preserves equality. This proves the sign identity. The order and accord identities follow from Theorem 2.4. □

Coordinate permutations act in the same way by permuting the entries of the sign vectors and the orientation indices. Together, coordinate reversals and permutations form the signed permutation group $C_2^d \rtimes S_d$. Thus the natural symmetry is larger than the antipodal duality alone.

8.2. **Projection.** For $I \subseteq [d]$, let π_I denote coordinate projection onto $\prod_{i\in I} X_i$. Sign representations commute with projection:

$$\chi(\pi_I x, \pi_I y) = \chi(x, y)|_I.$$

Proposition 8.2 (Compatibility under projection)**.** *If A is compatible about q in the full coordinate set, then $\pi_I(A)$ is compatible about $\pi_I(q)$ for every $I \subseteq [d]$.*

Proof. Restrict a compatible orientation σ to I. Every comparison with q remains valid after deleting coordinates. □

The converse fails: projection may delete a negative cycle. In the constraint representation, coordinate projection is vertex deletion after replacing each point-star by the equivalent signed clique on its support. This observation suggests feature-deletion problems governed by signed graph frustration.

9. Recognition, counting, and certificates

Let

$$L_q(A) = \sum_{x\in A} \max\{|S_q(x)| - 1, 0\}.$$

This is the number of signed edges in the sparse star representation of Theorem 4.3.

Algorithm 9.1 (Accord recognition and counting)**.** Given q and the support-sign pairs $(S_q(x), \alpha_q(x))$, $x \in A$:

1. For each x with $|S_q(x)| \geq 2$, select one root coordinate and add the signed star of Theorem 4.3.
2. In every unvisited connected component, choose an arbitrary root sign and propagate vertex signs by depth-first search.

3. If an edge disagrees with the propagated signs, return "incompatible" together with the negative cycle formed by that edge and the search-tree path between its endpoints.
4. Otherwise let c be the number of connected components and return

$$a_q(A) = 2^c, \qquad a_q(A)/2 = 2^{c-1}$$

for oriented and antipodal counts, respectively.

Theorem 9.2 (Complexity)**.** *Theorem 9.1 is correct and runs in*

$$O(d + L_q(A))$$

time and space once the support-sign representation is given. For dense numeric input, computing that representation and running the algorithm take $O(d|A|)$ *time, which is linear in the input size.*

Proof. Correctness follows from Theorems 4.4 and 4.5. The graph has d vertices and $L_q(A)$ edges. Construction and depth-first search are linear in the number of vertices and edges. A dense input contains $d|A|$ coordinate values relative to the anchor. □

The same test can be maintained incrementally by a disjoint-set data structure with parity labels. Each new constraint either merges two components, is redundant, or exposes a parity conflict. This gives near-linear online updates and permits the accord dimension

$$\dim_{\mathbb{F}_2} \mathrm{Acc}_q(A) = c_q(A)$$

to be maintained as points arrive.

10. Relation to order, cube, and skyline structures

The finite orthant-order system should be distinguished from several derived constructions.

(i) A cover graph or Hasse diagram records only cover relations in one quotient poset. It is a sparse representation of a single order, not the primary object here.

(ii) The skyline in one orientation is the set of maximal or minimal elements of one product order. A global skyline relative to a query point combines skyline sets across its orthants [17]. Orthant-order compatibility instead asks whether several points share at least one antipodal orientation pair; it does not select extrema.

(iii) The coordinate hyperplane arrangement supplies the sign-vector geometry $\{-, 0, +\}^d$ and chamber set $\{-, +\}^d$ [1]. A finite orthant-order system enriches this fixed arrangement by a data-indexed pairwise sign map.

(iv) Signed-graph balance supplies the exact consistency criterion [14, 22]. Here the signed graph is not given independently: it is functorially induced from anchored coordinate comparisons.

(v) Hypergraph DP-colouring can be expressed by avoiding prescribed partial Boolean colourings [4]. For two colours, the associated forbidden sets are Boolean faces, and antipodal face-pair coverings have been studied explicitly [20]. Anchored anti-accord is the corresponding avoidance problem for face pairs produced by point–anchor sign vectors; anchored accord is their common intersection problem.

(vi) Random high-dimensional product orders are usually studied through their height, width, or chain thresholds [7]. Orientation density instead records how a fixed data set is distributed over the complete family of coordinate reversals.

The relationship to [18] is particularly direct. That paper introduced coordinate-reversed order structures for local structure-aware learning and obtained the two-point multiplicity of Theorem 2.7. The present work moves the orientation family to the foreground, removes dependence on cover graphs, treats arbitrary point families, and replaces pairwise multiplicity by the exact affine and signed-graph count of Theorem 4.5. The incomparability theory adds the dual Boolean-face avoidance problem and separates its affine and NP-complete regimes.

11. Further directions

The present theory leaves several natural problems.

Enumeration. By Theorems 5.10 and 5.11, the homotopy type of $\mathcal{K}_q(D)$ is that of the nerve of a family of subcubes of the folded cube. Characterize the nerves, f-vectors, and homotopy types actually achievable by anchored data, and determine when all 2^{d-1} projective orientations give distinct facets. Even when all coordinates are chains, not every abstract simplicial complex can arise because the constraint sets have a binary affine representation.

Extremal accord. By Theorem 5.13, computing a largest compatible subset is NP-hard already for binary supports, where it coincides with max cut. Remaining questions concern the exact approximability for larger supports, in which one point bundles several signed edges, and the number and structure of maximal compatible subsets for fixed d, $n = |D|$, and q. The general-position case reduces to antipodal classes of full sign vectors, whereas ties create higher-dimensional Boolean faces.

Repair and frustration. Given an incompatible family, delete the fewest points or coordinates to make it compatible. The coordinate-deletion version is related to balancing a signed graph by vertex deletion; point deletion is a labelled-edge variant because one point may generate several constraints.

Antichain spectra and coverings. Improve the union bound in Theorem 6.9 by exploiting the intersection pattern of the antipodal faces $C(x, y)$.

Determine the possible values of $|\operatorname{Ant}(A)|$, characterize minimal cube coverings arising from point-pair sign maps, and compare anchored anti-accord coverings with general two-colour DP-covers.

Random thresholds. Theorem 6.12 concerns a *fixed* orientation. For the same continuous product model, [19, Theorem 9.3] treats the different question of whether *some* antichain orientation exists. Writing $\lambda_N = \binom{N}{2}2^{1-d}$, the latter critical window is $\lambda_N - (d-1)\ln 2 = O(1)$, rather than $\lambda_N = O(1)$. With ties, a single pair can cover several projective classes at once. How does their joint coverage change the threshold? Random signed-NAE thresholds for the anchored anti-accord $\operatorname{Inc}_q(A)$ are a separate question.

Dynamic systems. Maintain the accord count and a negative-cycle certificate under point insertions and deletions. Parity disjoint sets handle insertions; fully dynamic updates require a dynamic signed-connectivity structure.

Beyond the coordinate arrangement. The sign representation uses only the covector geometry $\{-,0,+\}^d$ of the coordinate hyperplane arrangement, with orientations as chambers. Replacing the Boolean arrangement by an arbitrary hyperplane arrangement, or by an oriented matroid [6], poses anchored accord and anti-accord problems over general tope sets. Which parts of the theory survive—the coset structure, the balance criterion, the exact Helly number, the Dowker duality—is open; the affine core exploits the group structure of Σ_d, which general tope sets lack.

Order-system morphisms. A category of finite orthant-order systems may take sign-preserving maps as strong morphisms and coordinatewise monotone maps as covariant morphisms. The useful definition should make projection, restriction, quotient by duplicate records, and signed permutation symmetries into canonical operations rather than introduce categorical language without additional structure.

Heterogeneous data and learning applications. Accord dimension, anti-accord density, antichain orientation counts, and negative-cycle certificates are possible order-only features for data analysis. As inputs to a learning model, they can encode neighbourhood structure without replacing nominal labels by arbitrary numerical ranks; whether they improve prediction is an empirical question beyond the present paper. Products mixing chains, flat categorical posets, taxonomies, and set-inclusion coordinates fit the extension in Section 7. Any learning application should specify whether an intrinsically incomparable coordinate is a hard veto, as in the strict product order, or whether a different missing-coordinate semantics is intended, and should be compared with skyline, Pareto, and rank-based methods.

12. Conclusion

A finite set in a product of d chains carries a canonical family of 2^d coordinate-reversed product preorders. The family is highly redundant but not trivial: its information is exactly the pairwise $\{-,0,+\}^d$-valued sign map. Around an anchor, simultaneous comparability becomes an affine parity problem. The corresponding signed constraint graph gives a complete structural answer:

$$\text{compatibility} \iff \text{balance}, \qquad |\operatorname{Acc}_q(A)| = 2^{\text{number of components}}.$$

Negative cycles provide certificates of failure and show that minimal obstructions may be arbitrarily large as the dimension grows, although for fixed dimension the Helly number is exactly d, dropping to two in general position. The accord is a coset of an explicit binary subspace, the balance condition is the vanishing of a class in H^1, and the whole anchored problem is a rank-one sign-completion problem for the anchored sign matrix. By Dowker duality, the compatibility complex has the homotopy type of a nerve of subcubes of the folded cube, bounded in terms of d alone; yet the extremal problem of a largest compatible subset is already max cut in disguise and hence NP-hard.

Incomparability is not merely the negation of this compatibility problem. Requiring every anchored comparison to fail produces signed NAE constraints: the binary-support case remains an affine balance problem, but ternary supports already yield NP-completeness and #P-complete counting, and mixed comparability and incomparability requirements obey the same dichotomy. Requiring all point pairs to be incomparable turns antichain orientations into the complement of a union of antipodal Boolean faces. Exact pair counts then give deterministic antichain orientations, while random product data become antichains near the natural high-dimensional scale $d \approx 2\log_2 N$, with a Poisson limit law and the explicit threshold profile $\exp(-2^{-t})$ whenever $d - 2\log_2 N \to t$.

Finally, products of posets separate this orientation-induced phenomenon from intrinsic coordinate incomparability, as occurs with nominal or hierarchical data. Together these results expose a tractable affine core, a nonlinear avoidance boundary, and an asymptotic antichain regime within one indexed order system. They require neither distances nor cover relations and provide a foundation for further enumerative, algorithmic, and data-analytic work.

Statements and Declarations

Competing interests. The author has no relevant financial or non-financial interests to disclose.

Author contributions. Wuming Pan: Conceptualization, methodology, formal analysis, investigation, writing—original draft, and writing—review and editing.

Data availability. No datasets were generated or analysed during the current study.

Use of generative-AI tools. An AI-assisted language and coding tool was used to help organize the manuscript, edit LaTeX, and check the exposition of proofs and references. The author independently verified the mathematical statements, proofs, computations, source attribution, and final text, and takes full responsibility for the manuscript.

References

[1] M. Aguiar and S. Mahajan. *Topics in Hyperplane Arrangements*, volume 226 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2017. https://doi.org/10.1090/surv/226.

[2] R. Apfelbaum, I. Ben-Dan, S. Felsner, T. Miltzow, R. Pinchasi, T. Ueckerdt, and R. Ziv. Points with large quadrant depth. *Journal of Computational Geometry*, 2(1):128–143, 2011. https://doi.org/10.20382/jocg.v2i1a7.

[3] R. Arratia, L. Goldstein, and L. Gordon. Two moments suffice for Poisson approximations: The Chen–Stein method. *The Annals of Probability*, 17(1):9–25, 1989. https://doi.org/10.1214/aop/1176991491.

[4] A. Bernshteyn and A. Kostochka. DP-colorings of hypergraphs. *European Journal of Combinatorics*, 78:134–146, 2019. https://doi.org/10.1016/j.ejc.2019.01.011.

[5] A. Björner. Topological methods. In R. L. Graham, M. Grötschel, and L. Lovász, editors, *Handbook of Combinatorics*, volume 2, pages 1819–1872. Elsevier, Amsterdam, 1995.

[6] A. Björner, M. L. Vergnas, B. Sturmfels, N. White, and G. M. Ziegler. *Oriented Matroids*, volume 46 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, 2 edition, 1999. https://doi.org/10.1017/CBO9780511586507.

[7] B. Bollobás and G. Brightwell. Random high-dimensional orders. *Advances in Applied Probability*, 27(1):161–184, 1995. https://doi.org/10.2307/1428102.

[8] S. Börzsönyi, D. Kossmann, and K. Stocker. The skyline operator. In *Proceedings of the 17th International Conference on Data Engineering*, pages 421–430. IEEE Computer Society, 2001. https://doi.org/10.1109/ICDE.2001.914855.

[9] N. Creignou and M. Hermann. Complexity of generalized satisfiability counting problems. *Information and Computation*, 125(1):1–12, 1996. https://doi.org/10.1006/inco.1996.0016.

[10] C. H. Dowker. Homology groups of relations. *Annals of Mathematics*, 56(1):84–95, 1952. https://doi.org/10.2307/1969768.

[11] M. R. Garey and D. S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W. H. Freeman, New York, 1979.

[12] M. R. Garey, D. S. Johnson, and L. Stockmeyer. Some simplified NP-complete graph problems. *Theoretical Computer Science*, 1(3):237–267, 1976. https://doi.org/10.1016/0304-3975(76)90059-1.

[13] M. X. Goemans and D. P. Williamson. Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming. *Journal of the ACM*, 42(6):1115–1145, 1995. https://doi.org/10.1145/227683.227684.

[14] F. Harary. On the notion of balance of a signed graph. *Michigan Mathematical Journal*, 2(2):143–146, 1953. https://doi.org/10.1307/mmj/1028989917.

[15] F. Harary and J. A. Kabell. A simple algorithm to detect balance in signed graphs. *Mathematical Social Sciences*, 1(1):131–136, 1980. https://doi.org/10.1016/0165-4896(80)90010-4.

[16] H. T. Kung, F. Luccio, and F. P. Preparata. On finding the maxima of a set of vectors. *Journal of the ACM*, 22(4):469–476, 1975. https://doi.org/10.1145/321906.321910.

[17] J. Liu, L. Xiong, J. Pei, J. Luo, and H. Zhang. Skyline diagram: Efficient space partitioning for skyline queries. *IEEE Transactions on Knowledge and Data Engineering*, 33(1):208–222, 2021. https://doi.org/10.1109/TKDE.2019.2923914.

[18] W. Pan. Fast generation of local hasse graphs for learning from structurally connected instances. In G. Wang, T. Li, J. W. Grzymala-Busse, D. Miao, A. Skowron, and Y. Yao, editors, *Rough Sets and Knowledge Technology*, volume 5009 of *Lecture Notes in Computer Science*, pages 548–555, Berlin, Heidelberg, 2008. Springer. https://doi.org/10.1007/978-3-540-79721-0_74.

[19] W. Pan. Learning Orthant Comparability: Proper–Improper Separation and Compression without Side Information. 1id1 author preprint, version 1, 2026. 1id1:core:20261120-0001@v1. https://1id1.net/id/core/20261120-0001/v1/.

[20] V. N. Potapov. DP-colorings of uniform hypergraphs and splittings of Boolean hypercube into faces. *The Electronic Journal of Combinatorics*, 29(3):P3.37, 2022. https://doi.org/10.37236/10550.

[21] T. J. Schaefer. The complexity of satisfiability problems. In *Proceedings of the Tenth Annual ACM Symposium on Theory of Computing*, pages 216–226, New York, 1978. Association for Computing Machinery. https://doi.org/10.1145/800133.804350.

[22] T. Zaslavsky. Signed graphs. *Discrete Applied Mathematics*, 4(1):47–74, 1982. https://doi.org/10.1016/0166-218X(82)90033-6.