

# MONOIDAL TOKEN SPACES: INITIAL SOURCES AND TENSOR ADJOINTS

WUMING PAN

**ABSTRACT.** A monoidal token space is the predicate category of a monoid-valued record functor, equipped with its tensor over coproduct. We study general record functors, free-monoid presentations, and the classical Token Space of words over a carrier and a fixed core. Tensoring by a fixed factor preserves every nonempty initial source exactly when left factors are unique and the product-image inclusion is cartesian. A free-group example separates these conditions. For free-monoid presentations whose generator functor preserves two specified coproduct pullback squares, tensoring preserves every one-arrow initial lift, so only uniqueness remains. Polynomial generator functors satisfy this hypothesis; freeness alone does not suffice. In classical Token Space, the criterion becomes unambiguity of a language product, and compatible factors have coprotopological forgetful functors. Independently, in every monoidal token space, tensoring has a right adjoint exactly for factors with empty carrier. These factors form a powerset quantale whose residuals act on the whole category. Ordered signatures delimit the constructions, and finite automata give effective tests for regular heaps.

## 1. Introduction

Monoidal token spaces organize records that can be composed and transported along changes of carrier. The classical Token Space [12] uses finite words over carrier elements and fixed core symbols. Its concatenation tensor raises a categorical question: when does composition with a fixed structure preserve initial lifts? We formulate Token Space at three levels and prove a general criterion, a transport theorem for free-monoid presentations, and a language criterion for the classical case.

Let  $\overline{F} : \mathbf{Set} \rightarrow \mathbf{Mon}$  have underlying set functor  $F$ . An object of  $\mathbf{Pred}(F)$  is a pair  $(A, L)$  with  $L \subseteq F(A)$ , and a morphism is a function  $f : A \rightarrow A'$  satisfying  $F(f)[L] \subseteq L'$ . The forgetful functor is topological. The monoid structures give a tensor over coproduct whose heap is

$$L \boxtimes K = \{F(\iota_A)(x)F(\iota_B)(k) : x \in L, k \in K\}.$$

It preserves final lifts in each variable and is an instance of the monoidal Grothendieck construction [11]. Preserving initial lifts requires both intersections within fibres and inverse images under changes of carrier to commute with tensoring.

We call this monoidal category a *monoidal token space*. The *free-monoid* level has  $F(A) = G(A)^*$  for a generator functor  $G : \mathbf{Set} \rightarrow \mathbf{Set}$ ; the *classical* level takes  $G(A) =$

---

2020 Mathematics Subject Classification: 18M05, 54B30, 68Q45.

Key words and phrases: Token Space, topological functor, initial source, cartesian natural transformation, monoidal category, formal language, residual.

© Wuming Pan, 2026. Permission to copy for private use granted.

$A \sqcup E$ . These extend the fixed-core terminology of [12] while retaining the standard notation  $\text{Pred}(F)$ . The middle level is substantive: suitable pullback preservation by  $G$  implies transport of initial lifts, whereas a free-monoid presentation alone does not.

1.1. MAIN RESULTS. Fix  $Y = (B, K)$ , and write  $\mu_{A,B}(x, k)$  for the displayed record product. Put  $S_A = F(A) \boxtimes K$ , the set of all possible product records. Our general criterion (Theorem 3.1) says that  $- \otimes Y$  preserves all nonempty initial sources if and only if:

- (i)  $\mu_{A,B}(x, k) = \mu_{A,B}(x', k')$  implies  $x = x'$  for every carrier  $A$ ;
- (ii) the inclusion  $S \hookrightarrow F(- \sqcup B)$  is a cartesian natural transformation, equivalently  $F(f \sqcup \text{id}_B)^{-1}[S_{A'}] = S_A$  for every  $f : A \rightarrow A'$ .

The two conditions define a natural partial left-factor map with a domain stable under inverse images. A free-group record functor satisfies the first condition for a unit heap but fails the second (Example 3.4): identifying carrier letters can create a product representation that did not exist before reindexing.

For a free-monoid presentation, Proposition 3.7 shows that every one-arrow initial lift is preserved if  $G$  preserves the two coproduct pullback squares in Equation (3.6). Thus cartesianity is automatic and only uniqueness remains. Polynomial generator functors satisfy this sufficient hypothesis. Taking  $G$  to be the underlying free-group functor gives a free-monoid counterexample to transport (Example 3.9).

For classical token structures, take  $F(A) = W_E(A) = (A \sqcup E)^*$ , where the core alphabet  $E$  is fixed. A morphism renames carrier letters and fixes the core. The tensor concatenates words over disjointly tagged carrier alphabets, sharing only the core letters. Every one-arrow initial lift is preserved. The general criterion therefore reduces to

$$- \otimes (B, K) \text{ preserves nonempty initial sources} \iff K \cap E^+ K = \emptyset.$$

The left-tensor condition is  $K \cap KE^+ = \emptyset$ . The right condition is equivalently unambiguity of the ordinary language product  $E^* K$  (Corollary 4.9). It forbids pairs  $v, cv \in K$ , with  $c$  a nonempty core word, rather than forbidding individual words from beginning with core letters.

The compatible factors form monoidal full subcategories for every  $F$  (Proposition 3.5). For words these subcategories are also closed under subheaps, carrier inverse images and nonempty initial lifts. If  $E \neq \emptyset$ , they are not topological over sets and are not closed under all final images (Theorem 4.13), but their forgetful functors are copretopological in the sense of Greve, Szgeti and Tholen (Proposition 4.14). Preservation of the empty initial source is different: only the unit factor preserves all initial sources in the word category (Corollary 4.11).

The right-adjoint question has a separate, uniform answer. Tensoring on either side by  $(B, K)$  has a right adjoint exactly when  $B = \emptyset$ . We construct the adjoints explicitly as universal residuals. These exponentiable factors form the unital quantale  $\mathcal{P}(F(\emptyset))$ ,

whose residuals act on the whole category (Theorem 6.1 and Proposition 6.2). For words this is the core-language quantale  $\mathcal{P}(E^*)$ .

Atomic reconstruction identifies the tensors under study (Theorem 2.4). Two supporting sections delimit their scope and make the word criterion effective: Section 5 classifies position-preserving tensors by graded monoids, and Section 7 supplies automata tests and failure witnesses.

**1.2. RELATION TO EARLIER WORK.** The predicate categories and their lift formulas belong to the standard theory of concrete categories [1]. Classical token structures [12] have a relational presentation indexed by word shapes. Coherent multiplication of these shapes supplies additional data. The global tensor is covered by the monoidal Grothendieck construction over a cocartesian base [11, Section 4 and Appendix A.2]; our reconstruction isolates the singleton-preservation hypothesis on powerset fibres.

Within each powerset fibre, preservation of nonempty intersections is the standard separation condition on singleton images. Our criterion combines it with cartesianity of the product-image inclusion. The free-group and word examples show that the two conditions are independent.

Free-monoid extension and pullback preservation by polynomial functors are standard [6]. Our transport proposition applies them to the coproduct inclusions and a fixed heap. The two squares recover the variable factor and the prescribed fixed factor separately, yielding a criterion over varying carriers.

At a fixed alphabet, language concatenation is Boolean-valued convolution on a discrete monoid, within the frameworks of Day [4] and Dongol, Hayes and Struth [5]. Regular-language operations and iteration are classical [10]. Unambiguous products are likewise standard; Pin and Weil [14, Section 7] study their marked forms in polynomial closure. Our word criterion connects unambiguity of the unmarked product  $E^*K$  with preservation of initial sources over varying carriers. For empty carrier it reduces to suffix-freeness, with the left-handed version reducing to prefix-freeness [2]. Bruggink, König and Küpper [3] study recognizable cospan languages, where multiple decompositions must be handled by automata. Here the categorical issue is compatibility of unique factors with carrier reindexing.

Closed tensors on topological categories have a separate history. Greve’s framework [7, pp. 100–102] assumes a unique structure on a singleton and tensors with cartesian underlying sets. His Extension Theorem 1.2 extends partial closed structures under density and compatibility hypotheses. Under the same singleton normalization, Kelly and Rossi [9, Proposition 3.1] show that closed tensors have cartesian underlying sets. Here the tensor lies over coproduct, and the empty and full heaps give distinct structures on a singleton. Greve, Szigeti and Tholen [8, Theorem 3] extend compatible partial tensors and hom functors using lifters and cosemifinal liftings over a symmetric monoidal closed base. We classify the factors of a fixed tensor; their notion of copretopologicity also describes our compatible-factor subcategories.

The adjoint classification concerns the coproduct tensor rather than cartesian exponentiability of relational structures [13]. The powerset quantale and its residuals are

classical; our result identifies its objects as precisely the ambient tensor-exponentiable factors.

1.3. ORGANIZATION. Sections 2 and 3 give the three levels, the general criterion and free-monoid transport. Section 4 develops classical Token Space and its compatible factors; Section 6 classifies tensor adjoints. The signature and automata sections have the supporting roles described above; only the latter assumes finite alphabets.

The Token Space programme uses records to connect data, computational descriptions and observations, including those of Transformer-based language models [12]. The results here concern composition of these record structures; compatible operational interpretations are a further direction.

## 2. Monoidal token spaces and atomic tensors

Let  $F : \mathbf{Set} \rightarrow \mathbf{Set}$  be a functor. Define  $\mathbf{Pred}(F)$  to have objects  $(A, L)$ , where  $L \subseteq F(A)$ , and arrows

$$f : (A, L) \longrightarrow (B, K) \quad \text{such that} \quad F(f)[L] \subseteq K.$$

We call the objects *token structures*, their subsets  $L$  *heaps*, and their arrows *token maps*. The forgetful functor  $U_F$  is faithful. Families are set-indexed unless stated otherwise. In the comparison with [8] in Proposition 4.14, we also allow class-indexed sources in the usual set/class convention; the intersections there are taken inside a fixed heap.

2.1. DEFINITION. [Three levels of record structure] A monoidal record functor is a functor  $\overline{F} : \mathbf{Set} \rightarrow \mathbf{Mon}$ , with underlying set functor  $F$ . Its monoidal token space is  $\mathbf{Pred}(F)$  equipped with the tensor of Equation (2.1) and unit  $(\emptyset, \{1_{\overline{F}(\emptyset)}\})$ . Let  $\mathcal{L} : \mathbf{Set} \rightarrow \mathbf{Mon}$  be the free-monoid functor. A free-monoid token space over  $G$  is supplied with a natural monoid isomorphism  $\overline{F} \cong \mathcal{L} \circ G$  for some  $G : \mathbf{Set} \rightarrow \mathbf{Set}$ . The classical Token Space  $\mathbf{TS}_E$  is the free-monoid token space over  $G_E(A) = A \sqcup E$ , for a fixed set  $E$ .

This terminology extends the fixed-core construction of [12]; the underlying predicate categories are standard. Theorem 2.4 establishes the tensor in the definition. The free level has the usual pointwise universal property: every natural transformation from  $G$  to the underlying set functor of  $\overline{H} : \mathbf{Set} \rightarrow \mathbf{Mon}$  extends uniquely to a monoid-valued natural transformation  $\mathcal{L}G \rightarrow \overline{H}$ . Freeness gives each component, and agreement on generators proves naturality. By Proposition 2.5, the extension induces a functor preserving tensor, unit and final lifts. In particular, the records of  $\mathbf{TS}_E$  are freely generated by  $A \sqcup E$ . This pointwise universal property does not assert that  $\mathbf{TS}_E$  generates all monoidal token spaces in a category of such spaces.

2.2. PROPOSITION. The functor  $U_F$  has initial and final lifts, given respectively by

$$\bigcap_i F(f_i)^{-1}[L_i], \quad \bigcup_j F(g_j)[K_j]$$

for functions  $f_i : A \rightarrow A_i$  and  $g_j : B_j \rightarrow B$ . In particular the initial object is  $O_F = (\emptyset, \emptyset)$ .

PROOF. A function  $h : C \rightarrow A$  preserves the first heap if and only if every  $f_i h$  preserves its target heap. A function  $k : B \rightarrow D$  preserves the second heap if and only if every  $kg_j$  does. These are exactly the respective lifting properties. For the empty families the heaps are  $F(A)$  and  $\emptyset$ . ■

Write  $f_! L = F(f)[L]$  and  $f^* K = F(f)^{-1}[K]$ . The functor  $A \mapsto \mathcal{P}(F(A))$ ,  $f \mapsto f_!$  is covariant, and  $\mathbf{Pred}(F)$  is its Grothendieck construction.

2.3. DEFINITION. *An atomic tensor over coproduct on  $\mathbf{Pred}(F)$  consists of a monoidal structure whose underlying tensor on carriers and maps is coproduct, and whose constraints lift the canonical constraints in  $\mathbf{Set}$ , such that:*

- (i) *each tensoring functor preserves every final lift, including empty sinks;*
- (ii) *the tensor of two singleton heaps is a singleton heap;*
- (iii) *the unit has the form  $(\emptyset, \{u_0\})$ .*

*All coproduct identifications are fixed throughout.*

2.4. THEOREM. [Atomic reconstruction] *For a fixed functor  $F$ , atomic tensors over coproduct on  $\mathbf{Pred}(F)$  are in bijection with lifts  $\overline{F} : \mathbf{Set} \rightarrow \mathbf{Mon}$  of  $F$ . If the multiplication and unit of  $\overline{F}(A)$  are  $m_A, u_A$ , the corresponding tensor heap is*

$$L \boxtimes K = \{m_{A \sqcup B}(F(\iota_A)x, F(\iota_B)y) : x \in L, y \in K\}. \quad (2.1)$$

PROOF. Given an atomic tensor, write its operation on heaps as  $\boxtimes_{A,B}$ . Final lifts of identity functions are unions. Hence the operation preserves arbitrary unions in each variable, including the empty union. Atomicity defines functions

$$\mu_{A,B} : F(A) \times F(B) \longrightarrow F(A \sqcup B), \quad \{x\} \boxtimes \{y\} = \{\mu_{A,B}(x, y)\},$$

and

$$L \boxtimes K = \bigcup_{x \in L, y \in K} \{\mu_{A,B}(x, y)\}.$$

Preservation of one-arrow final lifts in both variables gives

$$F(f \sqcup g)\mu_{A,B}(x, y) = \mu_{A',B'}(F(f)x, F(g)y). \quad (2.2)$$

The lifted associator and unitors give precisely the associativity and unit equations for  $\mu, u_0$ . Thus  $F$ , with these maps, is lax monoidal from  $(\mathbf{Set}, \sqcup, \emptyset)$  to  $(\mathbf{Set}, \times, 1)$ .

Let  $\nabla_A : A \sqcup A \rightarrow A$  be the fold and  $!_A : \emptyset \rightarrow A$  the unique map. Set

$$m_A = F(\nabla_A)\mu_{A,A}, \quad u_A = F(!_A)u_0.$$

Naturality follows from Equation (2.2). Both ways of multiplying three elements are the corresponding threefold  $\mu$ , followed by  $F$  of the threefold fold. The associativity of  $\mu$

identifies them. Its unit equations give the unit equations for  $m_A, u_A$ . These operations make every  $F(f)$  a monoid homomorphism.

Conversely suppose  $F$  has a lift to monoids. Define  $\mu_{A,B}$  by multiplying the two included elements as in Equation (2.1). Monoid multiplication and the naturality of the lift give the lax monoidal equations. Taking pointwise products of heaps gives a tensor with canonical constraints. Formula Equation (2.1) commutes with unions and with direct images, so it preserves final lifts. Its unit is  $(\emptyset, \{u_\emptyset\})$ , and it preserves singleton heaps.

The constructions are inverse. In one direction, naturality of  $m$  along  $\nabla_A$  recovers  $m_A$ . In the other, naturality of  $\mu$  along  $\iota_A, \iota_B$ , followed by the fold of  $(A \sqcup B) \sqcup (A \sqcup B)$ , recovers  $\mu_{A,B}$ . ■

**2.5. PROPOSITION.** [Compatibility with translations] *Let  $F, G$  carry the structures in Theorem 2.4. Natural transformations  $F \Rightarrow G$  whose components are monoid homomorphisms correspond to functors  $H : \mathbf{Pred}(F) \rightarrow \mathbf{Pred}(G)$  over  $\mathbf{Set}$  that preserve singleton heaps, all final lifts, and tensor and unit exactly.*

**PROOF.** A natural transformation  $\tau$  induces  $H(A, L) = (A, \tau_A[L])$  and  $H(f) = f$ . Naturality makes this well defined and makes it preserve final lifts; the monoid equations give tensor and unit preservation. Conversely, singleton preservation defines  $H(A, \{x\}) = (A, \{\tau_A(x)\})$ . Final lifts over identity maps determine  $H$  on all heaps. Final lifts along an arbitrary  $f$  give  $G(f)\tau_A = \tau_B F(f)$ . Tensor preservation gives compatibility with  $\mu$ , hence with multiplication after a fold. Unit preservation gives the units. ■

**2.6. REMARK.** The lax-monoidal/monoid-valued correspondence in this proof is the co-cartesian case of the monoidal Grothendieck construction [11]. The powerset calculation explains which additional atomicity hypotheses allow the monoid structure on records themselves to be recovered. Without atomicity, multiplication of records may be multi-valued, as in relational convolution [5].

**2.7. LEMMA.** [Disjoint singleton images] *Let  $T : \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$  preserve arbitrary unions. The following are equivalent:*

- (i)  *$T$  preserves binary intersections;*
- (ii)  *$T(\{x\}) \cap T(\{x'\}) = \emptyset$  whenever  $x \neq x'$ ;*
- (iii)  *$T$  preserves every nonempty intersection.*

*Equivalently, there is a function  $p : S \rightarrow X$ , with  $S = T(X) \subseteq Y$ , such that  $T(L) = \{y \in S : p(y) \in L\}$ . It preserves the empty intersection as well exactly when  $S = Y$ .*

**PROOF.** Binary intersection preservation and  $T(\emptyset) = \emptyset$  give disjointness. Conversely, disjointness assigns each  $y \in S$  a unique  $x$  whose singleton image contains it. Preservation of unions gives the stated formula. That formula preserves every nonempty intersection, and it preserves top exactly when  $S = Y$ . ■

### 3. Initial sources for fixed tensor factors

Let  $\overline{F} : \mathbf{Set} \rightarrow \mathbf{Mon}$ , with underlying functor  $F$ , and fix  $Y = (B, K)$  in  $\mathbf{Pred}(F)$ . Use the atomic tensor of [Theorem 2.4](#), with record multiplication

$$\mu_{A,B}(x, k) = F(\iota_A)(x)F(\iota_B)(k) \quad \text{in } F(A \sqcup B).$$

For  $x \in F(A)$  put

$$P_A(x) = \{\mu_{A,B}(x, k) : k \in K\}, \quad S_A = \bigcup_{x \in F(A)} P_A(x).$$

Thus  $S_A$  is the set of records that can be obtained by tensoring some left record with  $K$ , and the action on heaps is

$$T_A(L) = L \boxtimes K = \bigcup_{x \in L} P_A(x).$$

For a function  $f : A \rightarrow A'$ , write  $q_f = F(f \sqcup \text{id}_B)$ . Naturality gives

$$q_f(\mu_{A,B}(x, k)) = \mu_{A',B}(F(f)(x), k). \quad (3.1)$$

In particular  $q_f[S_A] \subseteq S_{A'}$ , so the sets  $S_A$  form a subfunctor of  $A \mapsto F(A \sqcup B)$ . Write  $j : S \hookrightarrow F(- \sqcup B)$  for the natural inclusion. A natural transformation is *cartesian* when all its naturality squares are pullbacks.

**3.1. THEOREM.** [General initial-source criterion] *The functor  $T_Y = - \otimes Y$ , over  $A \mapsto A \sqcup B$ , preserves every nonempty initial source if and only if both of the following conditions hold:*

(i) *For every  $A$ , the left factor is unique:*

$$\mu_{A,B}(x, k) = \mu_{A,B}(x', k') \implies x = x' \quad (x, x' \in F(A), k, k' \in K). \quad (3.2)$$

(ii) *The inclusion  $j : S \hookrightarrow F(- \sqcup B)$  is a cartesian natural transformation. Equivalently, product images are stable under inverse images: for every  $f : A \rightarrow A'$ ,*

$$q_f^{-1}[S_{A'}] = S_A. \quad (3.3)$$

*The corresponding left-tensor criterion uses  $\mu_{B,A}(k, x)$  and uniqueness of  $x$ .*

PROOF. Suppose  $T_Y$  preserves nonempty initial sources. Applying this to two identity carrier maps gives preservation of binary intersections by  $T_A$ . Since  $T_A$  preserves arbitrary unions, Lemma 2.7 says exactly that  $P_A(x)$  and  $P_A(x')$  are disjoint for  $x \neq x'$ . This is Equation (3.2). The one-arrow source  $f : A \rightarrow A'$  with target heap  $F(A')$  has initial heap  $F(A)$ . Preservation of this source gives Equation (3.3).

Conversely, assume the two conditions. Uniqueness defines a function  $p_A : S_A \rightarrow F(A)$  by  $p_A(\mu_{A,B}(x, k)) = x$ . For every heap  $L \subseteq F(A)$ ,

$$T_A(L) = \{z \in S_A : p_A(z) \in L\}. \quad (3.4)$$

By Equation (3.1),  $p_{A'}q_f(z) = F(f)p_A(z)$  for  $z \in S_A$ . Together with Equation (3.3), this proves, for every  $H \subseteq F(A')$ ,

$$q_f^{-1}[T_{A'}(H)] = T_A(F(f)^{-1}[H]). \quad (3.5)$$

Indeed, the condition on product images first ensures that an element on the left belongs to the domain  $S_A$  of  $p_A$ ; naturality then identifies its left factor.

Now let  $(f_i : A \rightarrow A_i, H_i)_{i \in J}$  be a structured source with  $J \neq \emptyset$ . Formula Equation (3.4) preserves nonempty intersections, so Equation (3.5) gives

$$\begin{aligned} \bigcap_{i \in J} q_{f_i}^{-1}[T_{A_i}(H_i)] &= \bigcap_{i \in J} T_A(F(f_i)^{-1}[H_i]) \\ &= T_A\left(\bigcap_{i \in J} F(f_i)^{-1}[H_i]\right). \end{aligned}$$

This is precisely preservation of the initial lift. The same proof with the variables reversed proves the left-handed assertion. ■

The first condition is the powerset separation property of Lemma 2.7; it defines a natural left-factor map  $p : S \rightarrow F$ . The second makes its domain inclusion  $j$  cartesian. Formula Equation (3.4) describes tensoring by inverse image along  $p_A$ , followed by inclusion into the full set of output records. Cartesianity is a property of  $j$ ; no preservation of pullbacks by the functors  $S$  or  $F$  is assumed.

**3.2. COROLLARY.** [When one-arrow initial lifts are preserved] *If  $T_Y$  preserves every one-arrow initial lift, then it preserves all nonempty initial sources if and only if Equation (3.2) holds for every carrier  $A$ .*

PROOF. The hypothesis gives Equation (3.5) and, on taking  $H = F(A')$ , gives Equation (3.3). Apply Theorem 3.1. ■

**3.3. COROLLARY.** [Including the empty source] *The functor  $T_Y$  preserves all initial sources, including the empty source, if and only if every record of  $F(A \sqcup B)$  has a product representation with a unique left factor, for every  $A$ . Equivalently, Equation (3.2) holds and  $S_A = F(A \sqcup B)$  for every  $A$ .*

PROOF. The empty initial source has heap  $F(A)$ , so its preservation is exactly the asserted equality for  $S_A$ . When all these equalities hold, Equation (3.3) is automatic. Apply Theorem 3.1 to the remaining nonempty sources. ■

For the constant one-element monoid functor and  $K = \{1\}$ , this criterion holds for every  $B$ . Thus the word-specific conclusion that only the unit preserves all initial sources does not hold for arbitrary record functors.

**3.4. EXAMPLE.** [Unique factors without inverse-image stability] Let  $F(A)$  be the free group on  $A$ , viewed as a monoid. Set  $B = \{b\}$  and  $K = \{1\}$ . For every  $A$ ,  $\mu_{A,B}(x, 1) = F(\iota_A)(x)$ , and  $F(\iota_A)$  is injective. Thus Equation (3.2) holds on every carrier, and  $S_A$  is the subgroup generated by  $A$ .

Take  $A = \{a_1, a_2\}$ ,  $A' = \{a\}$ , and let  $f$  identify  $a_1$  and  $a_2$ . The reduced word

$$z = ba_1a_2^{-1}b^{-1} \in F(A \sqcup B)$$

does not belong to  $S_A$ , whereas  $q_f(z) = baa^{-1}b^{-1} = 1 \in S_{A'}$ . Hence Equation (3.3) fails. More explicitly, for the target heap  $H = \{1\}$ , the record  $z$  lies in  $q_f^{-1}[T_{A'}(H)]$ , but not in  $T_A(F(f)^{-1}[H])$ , which is contained in  $S_A$ . Tensoring therefore fails to preserve even this one-arrow initial lift. Cancellation after identifying letters is the obstruction.

This example shows why left-factor uniqueness alone cannot replace Theorem 3.1. Conversely, inverse-image stability alone need not imply uniqueness: in the word case it holds for every factor, while Example 4.10 exhibits nonunique left factors.

**3.5. PROPOSITION.** [Composition of compatible factors] *For every  $\overline{F}$ , the full subcategory of factors  $Y$  for which  $- \otimes Y$  preserves nonempty initial sources is monoidal. The analogous left subcategory and the intersection of the two subcategories are monoidal as well.*

**PROOF.** Tensoring with the unit is naturally the identity. If  $- \otimes Y$  and  $- \otimes Z$  preserve nonempty initial sources, so does their composite. The associator identifies this composite with  $- \otimes (Y \otimes Z)$ , over the canonical associativity isomorphism of the base functors. Left tensoring is handled in the same way. The intersection inherits the property. ■

The additional closure under subheaps and carrier inverse images in Theorem 4.13 will use the word criterion explicitly.

**3.6. TRANSPORT FOR FREE-MONOID TOKEN SPACES.** Using the chosen natural monoid isomorphism in Definition 2.1, we identify  $\overline{F}$  with  $\mathcal{L} \circ G$  and write  $F(A) = G(A)^*$ . For  $f : A \rightarrow A'$ , put  $q = f \sqcup \text{id}_B$ . Consider the two squares in **Set**:

$$\begin{array}{ccc} A & \xrightarrow{\iota_A} & A \sqcup B \\ f \downarrow & & \downarrow q \\ A' & \xrightarrow{\iota_{A'}} & A' \sqcup B \end{array} \quad \begin{array}{ccc} B & \xrightarrow{\iota_B} & A \sqcup B \\ \text{id}_B \downarrow & & \downarrow q \\ B & \xrightarrow{\iota'_B} & A' \sqcup B. \end{array} \quad (3.6)$$

Both are pullbacks. The hypothesis below requires that they remain pullbacks after applying  $G$ , including when a carrier is empty. In particular it holds if  $G$  preserves all pullbacks along coproduct injections.

**3.7. PROPOSITION.** [Free-monoid transport] *Suppose  $G$  preserves the squares in Equation (3.6) for all  $f$  and  $B$ . Then, for every  $H \subseteq F(A')$  and  $K \subseteq F(B)$ ,*

$$F(q)^{-1}[H \boxtimes K] = F(f)^{-1}[H] \boxtimes K. \quad (3.7)$$

*Thus every right-tensoring functor preserves one-arrow initial lifts, and its product-image inclusion is cartesian. It preserves all nonempty initial sources if and only if left factors are unique as in Equation (3.2). The analogous assertions hold for left tensoring.*

**PROOF.** Write  $h^\sharp$  for the letterwise extension of a function  $h$  to words. The inclusion from right to left in Equation (3.7) follows by naturality. For the reverse, choose  $w \in G(A \sqcup B)^*$ ,  $u \in H$  and  $v \in K$  with

$$G(q)^\sharp(w) = G(\iota_{A'})^\sharp(u) G(\iota'_B)^\sharp(v).$$

Letterwise maps preserve length, so cut  $w$  after  $|u|$  letters. For each prefix letter and its corresponding letter of  $u$ , the left pullback square after applying  $G$  supplies a unique letter of  $G(A)$ . Together these form a word  $x$  with  $G(\iota_A)^\sharp(x)$  equal to the prefix and  $G(f)^\sharp(x) = u$ . For each suffix letter and its corresponding letter of  $v$ , the right square, whose left edge is  $\text{id}_{G(B)}$ , says that the suffix letter is precisely the image of that letter of  $v$ . Consequently  $w = G(\iota_A)^\sharp(x) G(\iota_B)^\sharp(v)$ , as required. The argument includes empty words.

Taking  $H = F(A')$  gives Equation (3.3). Corollary 3.2 gives the criterion for nonempty sources. For left tensoring, interchange the two summands and apply the same letterwise argument to a fixed prefix and a variable suffix.  $\blacksquare$

**3.8. EXAMPLE.** [Polynomial generator functors] Every functor of the form

$$G(A) = \coprod_{s \in D} A^{P_s}$$

satisfies the hypothesis of Proposition 3.7, with arbitrary sets  $D, P_s$ . Indeed, maps preserve the shape  $s$  and act coordinatewise. In the left square of Equation (3.6), a coordinate whose image lies in  $A'$  already lies in  $A$ ; in the right square, a coordinate whose image is a specified element of  $B$  is already that element. These observations give the unique lifts in both squares; nullary shapes cause no exception. This is a special case of the standard fact that polynomial functors preserve pullbacks [6, Proposition 1.16]. Examples include  $G(A) = A^2$  and  $G_E(A) = A \sqcup E$ . The underlying set functor of  $\mathcal{L}$  is itself polynomial [6, Example 1.9].

**3.9. EXAMPLE.** [Freeness without transport] The free-monoid hypothesis alone does not imply Equation (3.7). Let  $G(A)$  be the underlying set of the free group on  $A$ , and take  $F(A) = G(A)^*$ . Use  $A, A', B, f$  and the group element  $z = ba_1a_2^{-1}b^{-1}$  from Example 3.4. Brackets denote a one-letter outer word; the group identity 1 is a letter, distinct from the outer empty word  $\varepsilon$ . Set  $H = \{[1]\}$  and  $K = \{[z]\}$ . Then

$$F(q)([z]) = [1] \in H \boxtimes K, \quad [z] \notin F(f)^{-1}[H] \boxtimes K.$$

The second assertion holds because  $z$  is not in the subgroup generated by  $A$ ; hence  $[z]$  is not in the image of  $F(\iota_A)$ . This gives a free-monoid token space in which even a one-arrow initial lift is not preserved. The original free-group record functor of [Example 3.4](#), in contrast, has no free-monoid presentation: its value on a nonempty set has nonidentity invertible elements, while a free monoid has none.

The two-square condition is a sufficient hypothesis on generators, not a necessary-and-sufficient classification of generator functors. It makes the middle level of [Definition 2.1](#) useful without identifying freeness itself with stability under reindexing.

#### 4. Classical Token Space and concatenation

We now specialize the free-monoid level to  $G_E(A) = A \sqcup E$ .

Fix a set  $E$  and write  $W_E(A) = (A \sqcup E)^*$ . For  $f : A \rightarrow A'$ , write  $f^\# = W_E(f)$  for its letterwise extension fixing  $E$ . The category  $\mathbf{TS}_E$  has objects  $(A, L)$  with  $L \subseteq W_E(A)$ , called heaps, and morphisms  $f : (A, L) \rightarrow (A', H)$  satisfying  $f^\#[L] \subseteq H$ . Write  $f_!L = f^\#[L]$  and  $f^*H = (f^\#)^{-1}[H]$ . The star in  $f^*$  denotes inverse image, whereas  $L^*$  denotes Kleene closure.

We use the standard sense of a topological functor: every structured source has an initial lift, without a uniqueness assumption on structures over a singleton. A source  $f_i : A \rightarrow A_i$ , with heaps  $H_i \subseteq W_E(A_i)$ , has initial heap  $\bigcap_i f_i^*H_i$ . Indeed, for any  $(C, M)$ , a function  $h : C \rightarrow A$  preserves this heap if and only if every  $f_i h$  preserves  $H_i$ . Dually, a sink  $g_j : B_j \rightarrow B$ , with heaps  $L_j$ , has final heap  $\bigcup_j (g_j)_!L_j$ . We allow set-indexed sources and sinks, including empty ones. The empty source gives  $W_E(A)$ , and the empty sink gives the empty heap. The initial object is  $(\emptyset, \emptyset)$ . These are instances of the general formulas in [Proposition 2.2](#).

We identify the word alphabets associated with iterated coproducts by the canonical bijections. Carrier summands remain disjoint from each other and from  $E$ , even when the original sets share elements. For heaps  $L \subseteq W_E(A)$ ,  $K \subseteq W_E(B)$ , put

$$L \odot K = \iota_{A!}L \cdot \iota_{B!}K \subseteq (A \sqcup B \sqcup E)^*.$$

The dot on the right is ordinary concatenation within one alphabet;  $\odot$  records that two different carrier summands are used.

4.1. PROPOSITION. *The formulas*

$$(A, L) \otimes (B, K) = (A \sqcup B, L \odot K), \quad I = (\emptyset, \{\varepsilon\}), \quad f \otimes g = f \sqcup g$$

*define a monoidal category. The forgetful functor to  $(\mathbf{Set}, \sqcup, \emptyset)$  is strict monoidal for the canonical constraints. The tensor preserves all final lifts separately.*

PROOF. Extension of a carrier map to words commutes with concatenation:

$$(f \sqcup g)_!(L \odot K) = (f_!L) \odot (g_!K).$$

Associativity and units follow from concatenation; coherence follows from the faithful forgetful functor. For a structured sink,

$$\left( \bigcup_j (g_j)_! L_j \right) \odot K = \bigcup_j (g_j \sqcup \text{id}_B)_! (L_j \odot K).$$

This includes the empty union; the other variable is identical. ■

The fibre  $\mathcal{P}(W_E(A))$  is a unital quantale: multiplication distributes over arbitrary unions. Its direct-image maps are unital quantale homomorphisms. Taking their monoidal Grothendieck construction gives exactly [Proposition 4.1](#). Strictness of the forgetful functor means that it preserves the chosen constraints, not that coproduct is literally strictly associative.

**4.2. PROPOSITION.** *The concatenation tensor admits no braiding, for any core alphabet  $E$ .*

PROOF. Let  $D(A) = (A, \emptyset)$ . Every function induces a morphism between such objects, and  $D(A) \otimes D(B) = D(A \sqcup B)$ . A putative braiding therefore gives a natural family of bijections  $\beta_{A,B} : A \sqcup B \rightarrow B \sqcup A$ . At  $B = \emptyset$ , it is a natural automorphism of the identity functor of **Set**, hence the identity by testing maps from a singleton. Naturality with respect to  $\emptyset \rightarrow B$  forces every  $A$ -element to the  $A$ -summand of the target. The analogous argument for  $B$  forces  $\beta$  to be the canonical swap. The identity-carrier morphisms  $D(A) \rightarrow (A, L)$  force that same underlying swap on arbitrary heaps.

Take singleton carriers with heaps  $\{a\}$  and  $\{b\}$ . The swap sends their product word to  $a_{\text{right}}b_{\text{left}}$ , whereas the target heap consists of  $b_{\text{left}}a_{\text{right}}$ . It is not a token map. ■

**4.3. PROPOSITION.** *A token structure  $(A, L)$  admits an internal monoid structure for  $\otimes$  exactly when  $L$  is a submonoid of  $W_E(A)$ . The structure is unique when it exists, and all token maps between such objects are internal monoid homomorphisms. The resulting full subcategory is reflective, with reflector  $(A, L) \mapsto (A, L^*)$ .*

PROOF. The unit has the unique underlying function  $\emptyset \rightarrow A$ . The unit laws force the underlying multiplication  $A \sqcup A \rightarrow A$  to be the identity on both summands, hence the fold. These maps preserve the heaps exactly when  $\varepsilon \in L$  and  $LL \subseteq L$ . Conversely these conditions make the fold and unit token maps, and their algebraic laws hold on carriers. Every function commutes with folds. For any submonoid  $M$ ,

$$f^\sharp[L] \subseteq M \iff f^\sharp[L^*] \subseteq M.$$

This is the reflection, whose unit has identity carrier map. ■

4.4. REMARK. The cartesian product and  $\otimes$  have the canonical interchange

$$(X \times Y) \otimes (Z \times T) \longrightarrow (X \otimes Z) \times (Y \otimes T)$$

induced by tensoring projections. Together with the unique maps to the terminal object and the diagonal of  $I$ , this is the usual duoidal structure of a monoidal category with finite products. No preservation of cartesian products by  $\otimes$  is asserted.

4.5. INITIAL SOURCES AND COMPATIBLE FACTORS. Fix  $Y = (B, K)$  and let  $T_Y(X) = X \otimes Y$ . Preservation of an initial source always refers to the underlying functor  $A \mapsto A \sqcup B$ : the transformed carrier maps are  $f_i \sqcup \text{id}_B$ . This is a condition on structured sources, not a claim that  $T_Y$  preserves products or other limits.

4.6. LEMMA. [One source map] *For every  $f : A \rightarrow A'$ ,  $H \subseteq W_E(A')$ , and  $K \subseteq W_E(B)$ ,*

$$(f \sqcup \text{id}_B)^*(H \odot K) = (f^*H) \odot K. \quad (4.1)$$

*There is an analogous identity for left tensoring.*

PROOF. Apply Proposition 3.7 to  $G_E(A) = A \sqcup E$ , which satisfies its hypotheses by Example 3.8. ■

For  $w \in (A \sqcup B \sqcup E)^*$ , call a factorization  $w = uv$  *admissible for  $K$*  if  $u \in (A \sqcup E)^*$  and  $v \in K$ , using the summand inclusions. Different admissible cuts give different prefixes because words are finite and their lengths differ.

4.7. LEMMA. [Unique boundary] *The following are equivalent:*

- (i)  $K \cap E^+K = \emptyset$ ;
- (ii) *for every carrier  $A$ , every word in  $(A \sqcup B \sqcup E)^*$  has at most one admissible cut.*

PROOF. Suppose  $w = u_1v_1 = u_2v_2$  and  $|u_1| < |u_2|$ . There is a nonempty word  $c$  with  $u_2 = u_1c$  and  $v_1 = cv_2$ . The segment  $c$  belongs both to the left alphabet  $A \sqcup E$  and to the right alphabet  $B \sqcup E$ . Hence  $c \in E^+$ . Since  $v_1, v_2 \in K$ , the first condition rules this out. Equal-length cuts are identical.

Conversely, if  $v, cv \in K$  for  $c \in E^+$ , choose  $A = \emptyset$ . The word  $cv$  has the admissible cuts  $\varepsilon \mid cv$  and  $c \mid v$ . ■

4.8. COROLLARY. [Word initial-source criterion] *Fix  $Y = (B, K)$ . Regarding  $T_Y = - \otimes Y$  as a functor above  $A \mapsto A \sqcup B$ , so that  $UT_Y = (- \sqcup B)U$ , the following conditions are equivalent:*

- (i)  $T_Y$  preserves all nonempty initial sources;
- (ii) *for every  $A$ , the map  $L \mapsto L \odot K$  preserves binary intersections of heaps;*
- (iii) *that map preserves binary intersections on the empty carrier;*
- (iv)  $K \cap E^+K = \emptyset$ .

*Left tensoring  $Y \otimes -$  preserves all nonempty initial sources exactly when  $K \cap KE^+ = \emptyset$ .*

PROOF. By Lemma 4.6, every one-arrow initial lift is preserved. Hence Corollary 3.2 reduces preservation of nonempty initial sources to uniqueness of the left factor. For words, that is uniqueness of the admissible cut, characterized by Lemma 4.7. This proves (i) equivalent to (iv). The implication (i) to (ii) uses two identity carrier maps, and (ii) implies (iii). If (iv) fails, choose  $v, cv \in K$ ,  $c \in E^+$ . On the empty carrier the disjoint heaps  $L_1 = \{\varepsilon\}$ ,  $L_2 = \{c\}$  satisfy

$$cv \in (L_1 \odot K) \cap (L_2 \odot K),$$

contradicting (iii). Word reversal gives the left-handed result. ■

4.9. COROLLARY. [Unambiguous product] *For  $K \subseteq (B \sqcup E)^*$ , the following are equivalent:*

- (i)  $K \cap E^+K = \emptyset$ ;
- (ii) *the product  $E^*K$  is unambiguous, meaning that the multiplication map  $E^* \times K \rightarrow E^*K$  is injective;*
- (iii) *for every carrier  $A$ , the product of the included languages  $(A \sqcup E)^*$  and  $K$  is unambiguous.*

*The left-handed version uses  $KE^*$ .*

PROOF. Conditions (i) and (iii) are Lemma 4.7; (iii) implies (ii) by taking  $A = \emptyset$ . If  $v, cv \in K$  with  $c \in E^+$ , then  $cv$  has the two distinct factorizations  $(\varepsilon, cv)$  and  $(c, v)$ , so (ii) implies (i). ■

Unambiguous product is standard language-theoretic terminology; marked unambiguous products play a central role in [14, Section 7]. Here the product is unmarked. The corollary identifies the core condition with this familiar property of a particular pair of languages; no new class of codes is being postulated. The categorical assertion additionally requires the inverse-image identity of Lemma 4.6. For  $B = \emptyset$ , the right condition is suffix-freeness and the left condition is prefix-freeness. For nonempty  $B$ , neither condition forbids all extensions by carrier letters.

For example, any heap whose records begin with a carrier letter satisfies the right condition:

$$K \subseteq B(B \sqcup E)^*.$$

Every word of  $E^+K$  instead begins with a core letter. Requiring a carrier letter at the end similarly ensures the left condition. These are sufficient record-format conditions, not necessary ones.

4.10. EXAMPLE. [The smallest shared-core ambiguity] Let  $e \in E$ ,  $B = \emptyset$ , and  $K = \{\varepsilon, e\}$ . The output  $e$  has the two decompositions

| left heap         | prefix        | suffix in $K$ |
|-------------------|---------------|---------------|
| $\{\varepsilon\}$ | $\varepsilon$ | $e$           |
| $\{e\}$           | $e$           | $\varepsilon$ |

so multiplying by  $K$  fails to preserve the intersection of those two disjoint heaps. Nevertheless it preserves every one-arrow initial lift by Lemma 4.6, and every final lift by Proposition 4.1.

4.11. COROLLARY. [The empty-source distinction] *Either right or left tensoring by  $Y$  preserves all initial sources, including the empty source, if and only if  $Y = I$ .*

PROOF. For right tensoring the empty source requires

$$W_E(A) \odot K = W_E(A \sqcup B) \quad \text{for every } A.$$

If  $b \in B$ , take  $A = \{a\}$ . The word  $ba$  has no admissible cut, so  $B = \emptyset$ . The empty word on the right then forces  $\varepsilon \in K$ . By Corollary 4.8,  $K \cap E^+K = \emptyset$ , and every nonempty word in  $K \subseteq E^*$  would belong to  $E^+K$ . Thus  $K = \{\varepsilon\}$ . Conversely tensoring with  $I$  is naturally the identity. The left-handed proof uses  $ab$ , or follows by reversal. ■

4.12. SUBCATEGORIES OF COMPATIBLE FACTORS. Let  $\mathcal{R}_E$  be the full subcategory of  $\mathbf{TS}_E$  consisting of  $(B, K)$  with  $K \cap E^+K = \emptyset$ . Let  $\mathcal{L}_E$  be defined by  $K \cap KE^+ = \emptyset$ . These are respectively the right and left factors from Corollary 4.8; their intersection imposes both conditions.

4.13. THEOREM. [Monoidal subcategories of compatible factors] *The full subcategories  $\mathcal{R}_E, \mathcal{L}_E, \mathcal{R}_E \cap \mathcal{L}_E$  have the following properties.*

- (i) *They contain  $I$  and are closed under  $\otimes$ .*
- (ii) *They are closed under subheaps and carrier inverse images, and under nonempty initial lifts computed in  $\mathbf{TS}_E$ .*
- (iii) *If  $E \neq \emptyset$ , neither  $\mathcal{R}_E$  nor  $\mathcal{L}_E$ , nor their intersection, is closed under all final images. Their forgetful functors to **Set** are not topological.*

*If  $E = \emptyset$ , all three subcategories are  $\mathbf{TS}_E$ .*

PROOF. By Corollary 4.8, these are exactly the compatible factors classified by Proposition 3.5. That proposition gives the unit and tensor closure, including the intersection.

The conditions are inherited by subheaps. If  $cv, v \in f^*K$ ,  $c \in E^+$ , then  $cf^\sharp(v), f^\sharp(v) \in K$ , contradicting the right condition; the left condition is identical. An intersection of a nonempty family of such inverse images is a subheap of any one of them, proving initial-lift closure.

For final images choose  $e \in E$  and collapse  $\{b_1, b_2\}$  to  $\{b\}$ . The heap  $\{b_1, eb_2\}$  satisfies both conditions, but its image  $\{b, eb\}$  fails the right condition. Similarly  $\{b_1, b_2e\}$  has image  $\{b, be\}$ , failing the left condition. Finally, each singleton heap on the empty carrier satisfies both conditions. An initial lift of the empty source on that carrier would have to contain every such heap, in particular  $\{\varepsilon\}$  and  $\{e\}$ . Its heap would violate both conditions. Thus no such initial lift exists. When  $E = \emptyset$ ,  $E^+ = \emptyset$ , proving the last statement. ■

The weaker lifting property that remains has a standard name. For a faithful functor  $V : \mathcal{C} \rightarrow \mathbf{Set}$ , an object  $X$  and a family of functions  $f_i : VX \rightarrow VY_i$ , a *V-colifter* consists of a morphism  $e : X' \rightarrow X$  and morphisms  $\beta_i : X' \rightarrow Y_i$  with  $V\beta_i = f_iVe$ , universal among such data: every  $g : Z \rightarrow X$  and  $h_i : Z \rightarrow Y_i$  satisfying  $Vh_i = f_iVg$  admit a unique morphism  $t : Z \rightarrow X'$  with  $et = g$  and  $\beta_it = h_i$  for every  $i$ . The functor is *copretopological* if these colifters exist with  $Ve$  an isomorphism, including for the empty family. This is the dual discrete-family formulation of [8, p. 364]. Faithfulness ensures that the lifted maps respect every arrow of a given diagram. As in that reference, we allow class-indexed sources in this definition.

**4.14. PROPOSITION.** [Copretopological lifting] *The forgetful functors from  $\mathcal{R}_E$ ,  $\mathcal{L}_E$ , and  $\mathcal{R}_E \cap \mathcal{L}_E$  to  $\mathbf{Set}$  are copretopological, and are not topological if  $E \neq \emptyset$ . For an object  $X = (A, L)$  in one of these categories and functions  $f_i : A \rightarrow A_i$  to carriers of objects  $Y_i = (A_i, K_i)$ , a colifter over  $\text{id}_A$  is given by the heap*

$$H = L \cap \bigcap_i f_i^* K_i.$$

**PROOF.** Write  $V$  for the relevant forgetful functor. Since  $H \subseteq L$ , closure under subheaps places  $X' = (A, H)$  in the same category. The identity on  $A$  defines  $e : X' \rightarrow X$ , and the functions  $f_i$  define the required morphisms  $\beta_i : X' \rightarrow Y_i$ . If  $g : Z \rightarrow X$  and  $h_i : Z \rightarrow Y_i$  are as above, the image of the heap of  $Z$  under  $(Vg)^\sharp$  lies in  $L$  and in every  $f_i^* K_i$ , hence in  $H$ . Thus  $Vg$  defines the required morphism  $Z \rightarrow X'$ ; its uniqueness follows from faithfulness. For the empty family,  $H = L$  and  $e = \text{id}_X$ . For a class-indexed source, the distinct subsets  $L \cap f_i^* K_i$  form a subclass of  $\mathcal{P}(L)$ , hence a set. Their intersection inside  $L$  therefore defines  $H$ , and the same universal-property argument applies. The last assertion is [Theorem 4.13](#). ■

The given object  $X$  supplies an upper bound for this lifting construction. Copretopologicity therefore does not supply the unrestricted empty-source initial lift that fails above.

**4.15. EXAMPLE.** For  $e \in E$ , the singleton heap  $\{e\}$  belongs to  $\mathcal{R}_E \cap \mathcal{L}_E$ , whereas its Kleene closure  $\{e\}^*$  does not. Thus the monoid reflector of [Proposition 4.3](#) need not preserve either boundary condition.

## 5. Ordered signatures and graded monoids

This section delimits the ordered-signature setting in which the atomic tensors of [Theorem 2.4](#) preserve record positions.

An ordered finitary relational signature is a set  $\Sigma$  with  $\text{ar} : \Sigma \rightarrow \mathbb{N}$ , where each relation has its displayed ordered argument positions. Its record functor is

$$F_\Sigma(A) = \prod_{s \in \Sigma} A^{\text{ar}(s)}.$$

There is an isomorphism over **Set**

$$\text{Pred}(F_\Sigma) \cong \mathbf{Str}(\Sigma), \quad \mathcal{P}(F_\Sigma(A)) \cong \prod_{s \in \Sigma} \mathcal{P}(A^{\text{ar}(s)}).$$

A nullary relation contributes a truth value because  $A^0 = 1$ .

**5.1. DEFINITION.** *An atomic tensor on  $\text{Pred}(F_\Sigma)$  is position-preserving if the singleton records  $(s, a_1, \dots, a_m)$  and  $(t, b_1, \dots, b_n)$  have product whose tuple is exactly*

$$(\iota_A a_1, \dots, \iota_A a_m, \iota_B b_1, \dots, \iota_B b_n).$$

*The requirement concerns positions, including positions whose values coincide. Thus it excludes deleting, repeating or permuting positions.*

**5.2. THEOREM.** [Ordered signature classification] *Position-preserving atomic tensors on  $\mathbf{Str}(\Sigma)$  are in bijection with monoid structures  $(\Sigma, \mu, e)$  for which*

$$\text{ar}(e) = 0, \quad \text{ar}(\mu(s, t)) = \text{ar}(s) + \text{ar}(t).$$

*Equivalently, they are  $\mathbb{N}$ -graded monoid structures with the given grading.*

**PROOF.** By atomicity there is a singleton product  $\mu_{A,B}$  on records, natural under direct images as in [Equation \(2.2\)](#). For  $m = \text{ar}(s), n = \text{ar}(t)$ , evaluate it on the generic tuples on  $[m]$  and  $[n]$ . Its output has a definite symbol  $\mu(s, t)$  of arity  $m + n$ . Every tuple on arbitrary carriers is a direct image of a generic tuple, including nullary tuples and tuples with repeated entries. Naturality therefore forces the same output symbol  $\mu(s, t)$  for every such pair. The associator and unitors applied to generic tuples give the monoid equations. The atomic unit must be a nullary record  $e$ .

Conversely a graded monoid multiplies the symbols, while concatenation combines the ordered tuples. Extending over all pairs of records gives the heap product. Its naturality, union preservation and coherence give the required tensor by [Theorem 2.4](#). The constructions are inverse. ■

The grading condition is the familiar monoid condition for the Cauchy product of graded sets,

$$(\Sigma \star \Gamma)_n = \coprod_{p+q=n} \Sigma_p \times \Gamma_q.$$

The theorem identifies this algebraic data from the behaviour of singleton heaps and final lifts.

**5.3. COROLLARY.** *If  $\Sigma$  has bounded arity and contains a positive-arity symbol, then it admits no position-preserving atomic tensor.*

**PROOF.** A symbol  $s$  of arity  $r > 0$  would have powers of arities  $kr$  for all  $k \geq 1$ . ■

The conclusion applies in particular to finite signatures with a positive-arity relation. A signature consisting only of nullary relations is not excluded.

**5.4. EXAMPLE.** [A finite signature that discards positions] Take one nullary and one unary relation, so  $F(A) = \{\varepsilon\} \sqcup A$ . Give this set the multiplication

$$\varepsilon \diamond z = z, \quad a \diamond z = a \quad (a \in A).$$

It selects the first nonempty letter, is associative, and has unit  $\varepsilon$ . Every  $F(f)$  is a monoid homomorphism, so [Theorem 2.4](#) gives an atomic tensor preserving final lifts. The product of two unary records is still unary. Thus the positional hypothesis in [Corollary 5.3](#) cannot be removed. The empty-heap naturality argument from [Proposition 4.2](#) also shows that this tensor has no braiding.

**5.5. WORD SHAPES.** Let

$$S_E = (E \sqcup \{x\})^*, \quad \text{ar}(s) = \text{the number of occurrences of } x \text{ in } s.$$

The placeholder  $x$  is a new symbol, distinct from  $E$ . Replacing successive placeholders by the entries of a tuple gives a natural isomorphism

$$F_{S_E}(A) \cong (A \sqcup E)^*.$$

Word concatenation makes  $S_E$  a graded monoid, and its position-preserving tensor is the tensor of [Proposition 4.1](#). More generally, for a subset  $S \subseteq S_E$ , a tensor whose product shape is literal concatenation exists precisely when  $\varepsilon \in S$  and  $S$  is closed under concatenation. This statement specifies the multiplication, rather than merely requiring the set of arities to be closed under addition.

**5.6. PROPOSITION.** [A marked universal property] *The graded monoid  $S_E$ , with generators  $E$  of degree zero and  $x$  of degree one, is initial among triples  $(M, c, v)$ , where  $M$  is a graded monoid,  $c : E \rightarrow M_0$ , and  $v \in M_1$ . The induced map  $S_E \rightarrow M$  defines a carrier-preserving strict monoidal functor from  $\mathbf{TS}_E$  to  $\mathbf{Pred}(F_M)$  preserving all final lifts.*

**PROOF.** The free-monoid extension sends each  $e$  to  $c(e)$  and  $x$  to  $v$ . It is uniquely determined and degree-preserving. On records it translates the shape and leaves its ordered tuple unchanged. This is a natural transformation respecting the monoid structures, so [Proposition 2.5](#) applies. ■

The universal property is stated in the category of marked graded monoids, or equivalently in its presentation by these signatures and shape translations. It does not assert freeness among all topological categories with a tensor.

## 6. Right adjoints and the quantale of core factors

Tensoring by an empty-carrier factor has a right adjoint on the whole predicate category. We first construct these adjoints, then show that they exhaust the possibilities. The construction does not require the uniqueness or transport conditions of [Theorem 3.1](#).

**6.1. THEOREM.** [Tensor-exponentiable factors] *Let  $\bar{F} : \mathbf{Set} \rightarrow \mathbf{Mon}$  and give  $\mathbf{Pred}(F)$  the tensor of [Equation \(2.1\)](#). For  $Y = (B, K)$ , the following are equivalent:*

- (i)  $(-) \otimes Y$  has a right adjoint;
- (ii)  $B = \emptyset$ ;
- (iii)  $Y \otimes (-)$  has a right adjoint.

When  $B = \emptyset$ , write  $k_C = F(!_C)(k)$  for  $k \in K$ , where  $!_C : \emptyset \rightarrow C$ . The right adjoints are respectively

$$(C, H) \mapsto (C, H/K), \quad (C, H) \mapsto (C, K \setminus H),$$

where, using the monoid structure on  $F(C)$ ,

$$H/K = \{x : \forall k \in K, xk_C \in H\}, \quad K \setminus H = \{x : \forall k \in K, k_C x \in H\}.$$

**PROOF.** Suppose  $B = \emptyset$ . Under the canonical identification  $A \sqcup \emptyset \cong A$ , the heap of  $(A, L) \otimes Y$  consists of all  $xk_A$ , where  $x \in L, k \in K$ . For  $f : A \rightarrow C$ , naturality of multiplication gives

$$F(f)(xk_A) = F(f)(x)k_C.$$

Consequently  $f$  is a map from  $(A, L) \otimes Y$  to  $(C, H)$  if and only if  $F(f)[L] \subseteq H/K$ . The resulting hom-set bijection uses the same carrier functions. For a morphism  $h : (C, H) \rightarrow (C', H')$ , the same calculation gives  $F(h)[H/K] \subseteq H'/K$ . Thus the displayed formula defines a functor and the bijections are natural. The left-tensor adjoint is proved using  $k_A x$ .

Conversely, a functor with a right adjoint preserves the initial object  $O_F = (\emptyset, \emptyset)$ . Either tensoring functor sends  $O_F$  to an object with carrier  $B$  and empty heap. If  $B \neq \emptyset$ , this object has no map to  $O_F$ , and therefore is not initial. This proves necessity in both variables. ■

6.2. PROPOSITION. [Quantale of exponentiable factors] Put  $M = F(\emptyset)$ . The full subcategory of  $\text{Pred}(F)$  on the factors classified in [Theorem 6.1](#) is canonically the poset  $\mathcal{P}(M)$ , ordered by inclusion, with its powerset multiplication. It is a unital quantale and is left and right monoidal closed. The ambient right adjoints restrict to its quantale residuals.

For arbitrary  $(C, H)$  and  $K, L \subseteq M$ , the ambient residuals satisfy

$$\begin{aligned} H/(KL) &= (H/L)/K, \\ (KL) \backslash H &= L \backslash (K \backslash H), \\ (K \backslash H)/L &= K \backslash (H/L). \end{aligned}$$

Thus core factors act by residuated multiplication on the whole predicate category, as well as within the core fibre.

PROOF. By [Theorem 6.1](#), the objects are exactly  $(\emptyset, K)$ , with  $K \subseteq M$ . A morphism between two such objects is the unique empty function and exists precisely when the corresponding heaps are included. Their tensor is  $(\emptyset, KL)$ , with unit  $\{1_M\}$ . Arbitrary unions are joins and multiplication distributes over them. For core heaps, the two residuals obey

$$LK \subseteq H \iff L \subseteq H/K, \quad KL \subseteq H \iff L \subseteq K \backslash H.$$

They give the two closed structures on this poset. For a general carrier  $C$ , the map  $F(!_C)$  is a monoid homomorphism. The three identities follow by expanding the universal quantifiers and using associativity in  $F(C)$ . ■

The powerset quantale itself is classical. The proposition identifies it with exactly those objects whose tensoring functors have right adjoints in the ambient category. This is the precise sense of the closed part used here; it is not a classification of every full subcategory that might have internal adjoints of its own.

6.3. COROLLARY. [Language residuals] In  $\text{TS}_E$ , either tensoring functor associated with  $(B, K)$  has a right adjoint exactly when  $B = \emptyset$ . The full subcategory of these factors is the language quantale  $\mathcal{P}(E^*)$ . Their actions on arbitrary token structures have right adjoints with heaps

$$H/K = \{w \in W_E(C) : \forall v \in K, wv \in H\}, \quad K \backslash H = \{w \in W_E(C) : \forall v \in K, vw \in H\}.$$

The whole category  $(\text{TS}_E, \otimes)$  is neither left nor right monoidal closed.

PROOF. Apply [Theorem 6.1](#) and [Proposition 6.2](#) to  $W_E$ , for which  $W_E(\emptyset) = E^*$ . Objects with nonempty carriers exist, so not all tensoring functors have right adjoints. ■

The displayed operations are universal residuals, rather than existential language quotients sometimes denoted by the same slash. For example  $H/\emptyset = W_E(C)$  and  $H/\{\varepsilon\} = H$ .

6.4. EXAMPLE. [Independence of the two criteria] Let  $e \in E$ . The factor  $(\emptyset, \{\varepsilon, e\})$  has both right adjoints, but neither tensoring functor preserves every nonempty initial source. Conversely,  $(\{b\}, \{b\})$  satisfies both boundary conditions, but neither tensoring functor has a right adjoint.

## 7. Regular heaps and effective criteria

We now make [Corollary 4.8](#) effective for regular heaps, including witnesses of failed initial-source preservation. Throughout this section  $E$  and all carriers are finite. Regularity has its usual finite-alphabet meaning. Let  $\text{RegTS}_E$  be the full subcategory of  $\text{TS}_E$  on finite carriers with regular heaps.

**7.1. PROPOSITION.** *The category  $\text{RegTS}_E$  is closed under the tensor and the monoid reflector  $(A, L) \mapsto (A, L^*)$ . Initial lifts of finite sources and final lifts of finite sinks, over finite carriers, are regular. The two boundary criteria in [Corollary 4.8](#) characterize preservation of finite nonempty initial sources in this subcategory.*

**PROOF.** Renaming letters preserves regularity, as do concatenation, star, finite unions and finite intersections. Inverse images under letter maps are regular as well. These facts give the lift formulas. The sufficiency proof of [Corollary 4.8](#) applies to finite sources; its necessity uses only the regular singleton heaps  $\{\varepsilon\}$ ,  $\{c\}$ . ■

Arbitrary unions and intersections of regular languages need not be regular. No closure under arbitrary initial or final families is being asserted for this subcategory.

**7.2. A FINITE-STATE TEST WITH A BOUNDARY WITNESS.** Let  $\Sigma = B \sqcup E$ , and let  $\mathcal{A} = (Q, \Sigma, \delta, q_0, F_{\text{acc}})$  be a complete deterministic finite automaton for  $K$ , with  $n = |Q| \geq 1$ . Define

$$R_E = \{\delta(q_0, c) : c \in E^+\}.$$

This set is computed by reachability under  $E$ -labelled transitions, starting with all one-letter  $E$ -successors of  $q_0$ .

**7.3. THEOREM.** [Decision and witness bounds] *The condition  $K \cap E^+K = \emptyset$  holds if and only if no accepting pair in  $F_{\text{acc}} \times F_{\text{acc}}$  is reachable in the synchronous product automaton  $Q \times Q$  from the initial set*

$$\{(q_0, r) : r \in R_E\}.$$

*It can be decided in  $O(n + |E|n + |\Sigma|n^2)$  time, given the transition table. If it fails, one can produce  $c \in E^+$ ,  $v \in \Sigma^*$  with*

$$v, cv \in K, \quad 1 \leq |c| \leq n, \quad |v| \leq n^2 - 1.$$

*The left condition  $K \cap KE^+ = \emptyset$  is decidable in  $O(n + |\Sigma|n)$  time by testing whether a reachable accepting state has a positive-length  $E$ -path to an accepting state.*

**PROOF.** A right-condition violation is exactly a pair  $c, v$  such that

$$\delta(q_0, v) \in F_{\text{acc}}, \quad \delta(\delta(q_0, c), v) \in F_{\text{acc}}, \quad c \in E^+.$$

This is the displayed product-automaton reachability condition. The computation of  $R_E$  takes  $O(n + |E|n)$  time, and the product has  $n^2$  states and at most  $|\Sigma|n^2$  transitions.

Store shortest positive  $E$ -paths witnessing membership in  $R_E$ . Such a path has length at most  $n$ : a shortest positive path to  $q_0$  is a simple cycle, and a shortest path to another state is simple. A shortest product path to the accepting set has length at most  $n^2 - 1$ . Tracing the product path back selects its initial state and hence a stored core word  $c$ , giving the witness.

A left-condition violation has the form  $v, vc \in K$ , with  $c \in E^+$ . It exists precisely when a state in  $F_{\text{acc}}$ , reachable from  $q_0$ , has a positive  $E$ -path to  $F_{\text{acc}}$ . Ordinary forward reachability and reverse  $E$ -reachability, with one explicit first  $E$ -transition to enforce positivity, give the stated time bound. The conclusion includes the cases of empty language or empty core. ■

The returned  $c, v$  have an immediate categorical interpretation: they exhibit the failed intersection of  $(\emptyset, \{\varepsilon\})$  and  $(\emptyset, \{c\})$  after right tensoring by  $(B, K)$ . Thus the procedure produces a witness to failure of an initial lift, not just a Boolean answer.

**7.4. EFFECTIVE LANGUAGE RESIDUALS.** Let  $H \subseteq (C \sqcup E)^*$  be recognized by a complete DFA  $\mathcal{H} = (Q, C \sqcup E, \delta, q_0, F_H)$ , and let  $K \subseteq E^*$  be recognized by a complete DFA  $\mathcal{K} = (P, E, \eta, p_0, F_K)$ .

**7.5. PROPOSITION.** *The universal residual  $H/K$  is recognized by the same states, transitions and initial state as  $\mathcal{H}$ , with accepting set*

$$G = \{q \in Q : \forall v \in K, \delta(q, v) \in F_H\}.$$

*The set  $G$  is computable by reachability in  $Q \times P$ . The residual  $K \setminus H$  has a deterministic automaton with at most  $2^{|Q|}$  states. Consequently [Corollary 6.3](#) holds in  $\text{RegTS}_E$  as well.*

**PROOF.** A word  $w$  belongs to  $H/K$  exactly when  $\delta(q_0, w) \in G$ . A state  $q$  lies outside  $G$  precisely when, in the product restricted to letters of  $E$ , a state of  $(Q \setminus F_H) \times F_K$  is reachable from  $(q, p_0)$ . Reverse reachability from that bad set computes all such  $q$ .

For the left residual, compute the set

$$S = \{\delta(q_0, v) : v \in K\}$$

by forward reachability in  $Q \times P$  from  $(q_0, p_0)$ , retaining the first coordinates of states whose second coordinate belongs to  $F_K$ . Use the powerset automaton on  $Q$ , with initial state  $S$ , transition  $T \mapsto \{\delta(q, a) : q \in T\}$ , and accepting states the subsets of  $F_H$ . It accepts precisely the words  $w$  for which every  $v \in K$  satisfies  $vw \in H$ . The empty subset is accepting, as required when  $K = \emptyset$ .

Both constructions keep the carrier  $C$  finite and make the residual heaps regular. Hence the adjunctions restrict to the regular subcategory. The initial-object obstruction for nonempty carriers is unchanged. ■

**7.6. FINITE VERIFICATION.** The word-verification program checks bounded instances of the intersection criterion and the finite-signature counterexample, and checks the automaton decision procedures against bounded exhaustive witnesses for complete automata with at most three states on one carrier letter and one core letter. The bounds in [Theorem 7.3](#) make the latter a complete witness search for those automata. The script also validates every witness it returns. These computations are reproducibility checks; the structural results are established by the proofs above.

The general-verification program separately checks the criterion on finite natural relational systems over a category with one nonidentity arrow. It also checks the free-group counterexample, the unambiguous-product equivalence on bounded word heaps, and the residual identities on a finite noncommutative monoid. These are finite error checks, not proofs of the general statements.

A third program checks both generator pullback squares and the transport identity on singleton heaps for four polynomial functors, over bounded finite carriers and words of length at most two. It also verifies the outer free-monoid counterexample of [Example 3.9](#). The singleton identities extend to arbitrary heaps by unions; the finite carrier and word bounds remain limitations of the computation.

## 8. Further directions in Token Space

The general criterion separates two reasons for failure of initial source preservation. Multiple left factors cause a failure within a fibre. Failure of inverse-image stability creates decompositions after reindexing that were unavailable before reindexing. The free-monoid transport theorem supplies a sufficient condition for stability at the level of generators. Polynomial generators satisfy it, and classical Token Space reduces the remaining obstruction to factorization over the shared core. The two free-group examples show both that general monoidal record functors can fail stability and that an outer free-monoid presentation alone does not repair the failure.

This suggests two extensions. First, one can seek weaker hypotheses on a generator functor than preservation of both squares in [Equation \(3.6\)](#), or hypotheses on monoid-valued functors presented by generators and relations. For traces or other presentations by generators and relations, neither uniqueness nor stability should be inferred solely from the corresponding fact for free words. Second, multivalued record multiplication gives natural relational kernels rather than single-valued products. The powerset separation argument still applies, and the remaining question is when their product images and partial factor maps satisfy the required inverse-image identities.

The quantale  $\mathcal{P}(F(\emptyset))$  gives a complementary part of the structure: it records exactly the factors that have ambient right adjoints, without requiring unique factorization. Relating this residuated action to the compatible-factor subcategories is a separate problem from either classification. In the word case, change of core alphabet is another source of variation: carrier reindexing preserves the boundary conditions, whereas changing which letters are shared can change them.

For computational interpretations of Token Space, a next question is which carrier changes preserve both record factorization and the specified state transitions and observations. A compositional correctness result would require an interpretation respecting these structures in a concrete computational fragment.

## References

- [1] J. Adámek, H. Herrlich and G. E. Strecker. *Abstract and Concrete Categories: The Joy of Cats*. Wiley, 1990. Reprinted in *Reprints in Theory and Applications of Categories* 17 (2006), 1–507. [Open-access reprint](#).
- [2] J. Berstel, D. Perrin and C. Reutenauer. *Codes and Automata*. Encyclopedia of Mathematics and its Applications 129. Cambridge University Press, 2009. doi:10.1017/CBO9781139195768.
- [3] H. J. S. Bruggink, B. König and S. Küpper. Concatenation and other closure properties of recognizable languages in adhesive categories. *Electronic Communications of the EASST* 58 (2013), GT-VMT 2013. doi:10.14279/tuj.eceasst.58.854.
- [4] B. Day. On closed categories of functors. In *Reports of the Midwest Category Seminar IV*, Lecture Notes in Mathematics 137, Springer, 1970, 1–38. [Author text](#).
- [5] B. Dongol, I. J. Hayes and G. Struth. Convolution algebras: relational convolution, generalised modalities and incidence algebras. *Logical Methods in Computer Science* 17(1) (2021), 13:1–13:34. [Published text](#).
- [6] N. Gambino and J. Kock. Polynomial functors and polynomial monads. *Mathematical Proceedings of the Cambridge Philosophical Society* 154(1) (2013), 153–192. arXiv:0906.4931.
- [7] G. Greve. General construction of monoidal closed structures in topological, uniform and nearness spaces. In K. H. Kamps, D. Pumplün and W. Tholen, editors, *Category Theory*, Lecture Notes in Mathematics 962, Springer, 1982, 100–114. doi:10.1007/BFb0066889.
- [8] G. Greve, J. Szigeti and W. Tholen. Lifting tensorproducts along non-adjoint functors. *Cahiers de topologie et géométrie différentielle* 23(4) (1982), 363–378. [Published text](#).
- [9] G. M. Kelly and F. Rossi. Topological categories with many symmetric monoidal closed structures. *Bull. Austral. Math. Soc.* 31(1) (1985), 41–59. doi:10.1017/S0004972700002264.
- [10] D. Kozen. A completeness theorem for Kleene algebras and the algebra of regular events. *Information and Computation* 110(2) (1994), 366–390. Earlier version: Cornell technical report TR90-1123, 1990. [Technical report](#).
- [11] J. Moeller and C. Vasilakopoulou. Monoidal Grothendieck construction. *Theory and Applications of Categories* 35 (2020), 1159–1207. [Journal page](#). See also arXiv:1809.00727.
- [12] W. Pan. Token Space: A Category Theory Framework for AI Computations. arXiv:2404.11624v4, 2026. First version, 2024.
- [13] J. Parker. Exponentiability in categories of relational structures. *Theory and Applications of Categories* 39 (2023), 493–518. [Journal page](#).
- [14] J.-E. Pin and P. Weil. Polynomial closure and unambiguous product. *Theory of Computing Systems* 30 (1997), 383–422. doi:10.1007/BF02679467.

College of Computer Science, Sichuan University  
 Chengdu, P.R. China, 610065  
 Email: panwuming@scu.edu.cn